Abstract

In this paper we introduce the notion of Boolean filters in a pseudo-complemented distributive lattice and characterize the class of all Boolean filters. Further a set of equivalent conditions are derived for a proper filter to become a prime Boolean filter. Also a set of equivalent conditions is derived for a pseudo-complemented distributive lattice to become a Boolean algebra. Finally, a Boolean filter is characterized in terms of congruences.

Keywords: Pseudo-complemented distributive lattice, Boolean algebra, Boolean filter, maximal filter, congruence.

AMS Subject Classification(2010): 06A06, 54A10.
Definition 1.1. [4] An algebra \((L, \land, \lor)\) of type \((2, 2)\) is called a lattice if for all \(x, y, z \in L\), it satisfies the following properties.

1. \(x \land x = x, x \lor x = x\)
2. \(x \land y = y \land x, x \lor y = y \lor x\)
3. \((x \land y) \land z = x \land (y \land z), (x \lor y) \lor z = x \lor (y \lor z)\)
4. \((x \land y) \lor x = x, (x \lor y) \land x = x\)

Definition 1.2. [4] A lattice \(L\) is called distributive if for all \(x, y, z \in L\) it satisfies the following properties.

1. \(x \land (y \lor z) = (x \land y) \lor (x \land z)\)
2. \(x \lor (y \land z) = (x \lor y) \land (x \lor z)\)

Definition 1.3. [4] Let \((L, \land, \lor)\) be a lattice. A partial ordering relation \(\leq\) is defined on \(L\) by \(x \leq y\) if and only if \(x \land y = x\) and \(x \lor y = y\).

The pseudo-complement \(b^*\) of an element \(b\) is the greatest element disjoint from \(b\), if such an element exists. The defining property of \(b^*\) is:

\[ a \land b = 0 \iff a \land b^* = a \iff a \leq b^* \]

where \(\leq\) is a partial ordering relation on the lattice \(L\).

A distributive lattice \(L\) in which every element has a pseudo-complement is called a pseudo-complemented distributive lattice. For any two elements \(a, b\) of a pseudo-complemented lattice, we have the following.

1. \(a \leq b\) implies \(b^* \leq a^*\)
2. \(a \leq a^{**}\)
3. \(a^{***} = a^*\)
4. \((a \lor b)^* = a^* \land b^*\)
5. \((a \land b)^{**} = a^{**} \land b^{**}\)

An element \(a\) of \(L\) is called a dense element if \(a^* = 0\) and the set \(D\) of all dense elements of \(L\) forms a filter in \(L\).

A proper filter \(P\) of a lattice \(L\) is called a prime filter if \(x \lor y \in P\) implies \(x \in P\) or \(y \in P\) for all \(x, y \in L\). A proper filter \(M\) of \(L\) is called maximal if there exists no proper filter \(Q\) such that \(M \subset Q\). In a distributive lattice, every maximal filter is a prime filter but not the converse. However, in a relatively complemented lattice, every prime filter is maximal. It is noted that prime filters have also been used to classify the 0-distributivity of semilattices [6]. For distributive lattices, we have the following theorem related to prime filters.

Theorem 1.4. [4] Let \(L\) be a distributive lattice and \(x, y \in L\) such that \(x \neq y\). Then there exists a prime filter \(P\) such that \(x \in P\) and \(y \notin P\).

Throughout this note, unless otherwise mentioned, all lattices are bounded and pseudo-complemented distributive lattices.
2 Boolean filters and their properties

In this section, the concept of Boolean filters is introduced in a pseudo-complemented distributive lattice. Further the direct products and the homomorphic images of Boolean filters are studied. Finally, a set of equivalent conditions are derived for every filter of $L$ to become a Boolean filter.

Definition 2.1. Let $L$ be a pseudo-complemented distributive lattice. A filter $F$ of $L$ is called a Boolean filter if $x \lor x^* \in F$ for each $x \in L$.

Since $x \lor x^* \in D$ for all $x \in L$, it is evident that $D$ is a Boolean filter of $L$. In fact it is the smallest Boolean filter of $L$.

Example 2.2. Let $L = \{0, a, b, c, d, 1\}$ be a distributive lattice whose Hasse diagram is given in the following figure.

![Hasse diagram of the distributive lattice $L = \{0, a, b, c, d, 1\}$](image)

Consider the filters $F_1 = \{a, c, d, 1\}; F_2 = \{b, c, d, 1\}; F_3 = \{c, d, 1\}; F_4 = \{d, 1\}$ and $F_5 = \{1\}$. Then clearly $F_1, F_2$ and $F_3$ are Boolean filters where as $F_4$ and $F_5$ are not Boolean, because of $a \lor a^* = a \lor b = c \not\in F_4 \cup F_5$.

Proposition 2.3. Every maximal filter of $L$ is a Boolean filter.

Proof. Let $M$ be a maximal filter of $L$. Suppose $x \lor x^* \not\in M$ for some $x \in L$. Then $M \lor [x \lor x^*] = L$. Hence $0 = a \land b$ for some $a \in M$ and $b \in [x \lor x^*]$. Then we have the following consequence.

\[
a \land b = 0 \Rightarrow a \land (x \lor x^*) = 0 \\
\Rightarrow a \land x = 0 \text{ and } a \land x^* = 0 \\
\Rightarrow a \leq x^* \text{ and } a \leq x^{**} \\
\Rightarrow a \leq x^* \land x^{**} = 0
\]

which is a contradiction to the fact that $0 \in M$. Hence $x \lor x^* \in M$ for all $x \in L$. Therefore, $M$ is a Boolean filter of $L$. 

Corollary 2.4. A proper filter of a pseudo-complemented lattice $L$ which contains either $x$ or $x^*$ for all $x \in L$ is a Boolean filter.
Proof. Let \( F \) be a proper filter of \( L \) satisfying the given condition. We show that \( F \) is maximal. Suppose \( G \) is a proper filter of \( L \) such that \( F \subset G \). Choose \( a \in G - F \). Since \( a \notin F \), by the condition, we get \( a^* \in F \subset G \). Since \( a \in G \) and \( a^* \in G \), we get \( 0 = a \land a^* \in G \), which is a contradiction. Therefore, \( F \) is a maximal filter. Thus by Proposition 2.3, \( F \) is a Boolean filter. \( \blacksquare \)

Corollary 2.5. In a relatively complemented lattice, every prime filter is a Boolean filter.

The converse of Proposition 2.3 is not true in general. For, in Example 2.2, the filter \( F_3 \) is a Boolean filter but not a maximal filter.

A set of equivalent conditions are derived for a Boolean filter to become a maximal filter.

Theorem 2.6. Let \( F \) be a proper filter of a pseudo-complemented lattice \( L \). Then the following conditions are equivalent.

1. \( F \) is maximal.
2. \( x \notin F \) implies \( x^* \in F \) for all \( x \in L \).
3. \( F \) is prime Boolean.

Proof. (1) \( \Rightarrow \) (2): Assume that \( F \) is a maximal filter of \( L \). Suppose \( x \in L - F \), then \( F \lor [x] = L \) which yields that \( a \land x = 0 \) for some \( a \in F \). Hence \( a \leq x^* \), which implies that \( x^* \in F \).

(2) \( \Rightarrow \) (3): Let \( x \in L \). Suppose \( x \lor x^* \notin F \). Then it is clear that \( x \notin F \) and \( x^* \notin F \), which is a contradiction to the condition (2). Hence \( F \) is a Boolean filter of \( L \). Suppose \( x \lor y \in F \) and \( x \notin F \). Then by condition (2), we get \( x^* \in F \). Hence \( x^* \land y = 0 \lor (x^* \land y) = (x^* \land x) \lor (x^* \land y) = x^* \land (x \lor y) \in F \). Since \( x^* \land y \leq y \), we get that \( y \in F \). Therefore, \( F \) is a prime Boolean filter of \( L \).

(3) \( \Rightarrow \) (1): Assume that \( F \) is a prime Boolean filter of \( L \). Suppose \( F \) is not maximal. There exists a proper filter \( F' \) of \( L \) such that \( F \subset F' \). Choose \( x \in F' - F \). Since \( F \) is Boolean, we get \( x \lor x^* \in F \). Since \( F \) is prime and \( x \notin F \), we get \( x^* \in F \subset F' \). Hence it concludes that \( 0 = x \land x^* \in F' \), which is a contradiction. Therefore, \( F \) is a maximal filter. \( \blacksquare \)

The following proposition is obvious from Definition 2.1.

Proposition 2.7. Let \( F, G \) be two filters of a pseudo-complemented lattice such that \( F \subset G \). If \( F \) is a Boolean filter then so is \( G \).

We now characterize the Boolean filters in the following:

Theorem 2.8. Let \( F \) be a proper filter of a pseudo-complemented lattice \( L \). Then the following conditions are equivalent.

1. \( F \) is a Boolean filter.
2. \( x^{**} \in F \) implies \( x \in F \).
3. For \( x, y \in L \), \( x^* = y^* \) and \( x \in F \) imply \( y \in F \).

Proof. (1) \( \Rightarrow \) (2): Assume that \( F \) is a Boolean filter of \( L \). Suppose \( x^{**} \in F \). Since \( F \) is a Boolean
filter, we get \( x \lor x^* \in F \). Hence \( x = x \lor 0 = (x \land x^*) \lor (x^* \land x^*) = (x \lor x^*) \land x^* \in F \). Therefore, condition (2) holds.

(2) \( \Rightarrow \) (3) : Let \( x, y \in L \) and \( x^* = y^* \). Suppose \( x \in F \), then \( y^* = x^* \in F \). Hence by the condition (2), it follows that \( y \in F \).

(3) \( \Rightarrow \) (1) : Let \( x \in D \). Then \( x^* = 0 \leq a^* \) for any \( a \in F \). Hence \( a^{**} \leq x^{**} \) and \( a^{**} \in F \). Hence \( x^{**} \in F \). Since \( x^* = x^{**} \) and \( x^{**} \in F \), by the condition (3), we get \( x \in F \). Hence \( D \subseteq F \). Since \( D \) is a Boolean filter, by Proposition 2.7, we get that \( F \) is a Boolean filter of \( L \).

Now we discuss about the homomorphic images of Boolean filters of pseudo-complemented distributive lattices. By a homomorphism on a pseudo-complemented lattice, we mean a bounded homomorphism which also preserves the pseudo-complementation, that is, \( f(x^*) = f(x)^* \) for all \( x \in L \).

**Theorem 2.9.** Let \( (L, \lor, \land, ^*, 0, 1) \) and \( (L', \lor, \land, ^*, 0', 1') \) be two pseudo-complemented lattices and \( \psi \) a homomorphism from \( L \) onto \( L' \). Then we have the following conditions.

1. \( \psi(F) \) is a Boolean filter of \( L' \) whenever \( F \) is a Boolean filter of \( L \).
2. \( \psi^{-1}(G) \) is a Boolean filter of \( L \) whenever \( G \) is a Boolean filter of \( L' \).

**Proof.** (1). Suppose \( F \) is a Boolean filter of \( L \). It is known that \( \psi(F) \) is a filter of \( L' \). Let \( y \in L' \). Since \( \psi \) is onto, there exists \( x \in L \) such that \( \psi(x) = y \). Since \( F \) is a Boolean filter of \( L \), we get \( x \lor x^* \in F \). Now \( y \lor y^* = \psi(x) \lor \psi(x)^* = \psi(x) \lor \psi(x^*) = \psi(x \lor x^*) \in \psi(F) \). Therefore, \( \psi(F) \) is a Boolean filter of \( L' \).

(2). Let \( G \) be a Boolean filter of \( L' \). Clearly \( \psi^{-1}(G) \) is a filter of \( L \). Let \( x \in L \). Then \( \psi(x \lor x^*) = \psi(x) \lor \psi(x^*) = \psi(x)^* \in G \), since \( \psi(x) \in L' \). Hence we get \( x \lor x^* \in \psi^{-1}(G) \). Therefore, \( \psi^{-1}(G) \) is a Boolean filter of \( L \).

Let \( L_1 \) and \( L_2 \) be two pseudo-complemented distributive lattices with \( * \) as their pseudo-complementation. Then \( L_1 \times L_2 \) is also a pseudo-complemented distributive lattice with respect to the point-wise operations in which the pseudo-complementation is given as follows:

\[ (a, b)^* = (a^*, b^*) \]

Now we discuss about the direct products of Boolean filters of a pseudo-complemented distributive lattice.

**Theorem 2.10.** If \( F_1 \) and \( F_2 \) are Boolean filters of \( L_1 \) and \( L_2 \) respectively, then \( F_1 \times F_2 \) is a normal filter of the product lattice \( L_1 \times L_2 \). Conversely, every Boolean filter \( F \) of \( L_1 \times L_2 \) can be expressed as \( F = F_1 \times F_2 \) where \( F_1 \) and \( F_2 \) are Boolean filters of \( L_1 \) and \( L_2 \) respectively.

**Proof.** Let \( F_1 \) and \( F_2 \) be Boolean filters of \( L_1 \) and \( L_2 \) respectively. Since \( 1 \in F_1 \) and \( 1 \in F_2 \), we get \( (1, 1) \in F_1 \times F_2 \). Clearly \( F_1 \times F_2 \) is a filter of \( L_1 \times L_2 \). Let \( x \in L_1 \) and \( y \in L_2 \). Since \( F_1 \) and \( F_2 \) are Boolean filters of \( L_1 \) and \( L_2 \) respectively, we get \( x \lor x^* \in F_1 \) and \( y \lor y^* \in F_2 \). Hence \( (x, y) \lor (x, y)^* = (x \lor x^*, y \lor y^*) \in F_1 \times F_2 \). Therefore, \( F_1 \times F_2 \) is a Boolean filter of \( L_1 \times L_2 \).
Conversely, let $F$ be any Boolean filter of $L_1 \times L_2$. Consider the projections $\Pi_i : L_1 \times L_2 \rightarrow L_i$ for $i = 1, 2$. Let $F_1$ and $F_2$ be the projections of $F$ on $L_1$ and $L_2$ respectively. That is $\Pi_i(F) = F_i$ for $i = 1, 2$. We prove that $F_1$ and $F_2$ are Boolean filters of $L_1$ and $L_2$ respectively. Since $(1, 1) \in F$, we get $1 = \Pi_1(1, 1) \in F_1$. Clearly $F_1$ is a filter of $L_1$. Let $x \in L_1$ and $x^{**} \in F_1$. Then $(x, 1)^{**} = (x^{**}, 1^{**}) = (x^{**}, 1) \in F$. Since $F$ is a Boolean filter, we get $(x, 1) \in F$. Thus $x = \Pi_1(x, 1) \in \Pi_1(F) = F_1$. Therefore, $F_1$ is a Boolean filter of $L_1$. Similarly, we get $F_2$ is a Boolean filter of $L_2$.

Next we prove that $F = F_1 \times F_2$. Clearly $F \subseteq F_1 \times F_2$. Let $(x, y) \in F_1 \times F_2$. Then $x^{**} \in F_1 = \Pi_1(F)$ and $y^{**} \in F_2 = \Pi_2(F)$. Hence $(x^{**}, 1) \in F$ and $(1, y^{**}) \in F$. Since $F$ is a filter, we have $(x, y)^{**} = (x^{**}, y^{**}) = (x^{**} \land 1, 1 \land y^{**}) = (x^{**}, 1) \land (1, y^{**}) \in F$. Since $F$ is a Boolean filter, we get that $(x, y) \in F$. Thus we have $F_1 \times F_2 \subseteq F$ and hence $F = F_1 \times F_2$.

We recall the well known Glivinko type congruence $\psi$ defined on $L$ such that $(x, y) \in \psi$ if and only if $x^* = y^*$ for all $x, y \in L$. We derive a set of equivalent conditions for every filter of $L$ to become a Boolean filter.

**Theorem 2.11.** Let $L$ be a pseudo-complemented distributive lattice. Then the following conditions are equivalent.

1. $L$ is a Boolean algebra.
2. Every filter is a Boolean filter.
3. Every principal filter is a Boolean filter.
4. Every prime filter is a Boolean filter.
5. $\psi$ is the smallest congruence.

**Proof.** (1) $\Rightarrow$ (2): It is a fact that $L$ is a Boolean algebra if and only if it has a unique dense element. Assume that $L$ has a unique dense element, precisely 1. Let $F$ be a filter of $L$. Then $x \lor x^* = 1 \in F$ for all $x \in L$. Therefore, $F$ is a Boolean filter of $L$.

(2) $\Rightarrow$ (3): It is obvious.

(3) $\Rightarrow$ (4): Assume that every principal filter of $L$ is a Boolean filter. Then clearly [1] is a Boolean filter of $L$. Since [1] $\subseteq P$, by Proposition 2.7, we get that $P$ is also a Boolean filter of $L$.

(4) $\Rightarrow$ (5): Assume that every prime filter is a Boolean filter. Let $x, y \in L$ be such that $(x, y) \in \psi$. Suppose $x \neq y$. Then there exists a prime filter $P$ such that $x \in P$ and $y \notin P$. Hence $y^{**} = x^{**} \in P$. Since $P$ is Boolean, we get $y \in P$, which is a contradiction. Hence $x = y$. Therefore, $\psi$ is the smallest congruence.

(5) $\Rightarrow$ (1): Assume the condition (5). Suppose $L$ has two dense elements, say $x, y$. Then we get $x^* = 0 = y^*$. Hence $(x, y) \in \psi$.

Therefore, by condition (5), we get $x = y$. Thus $L$ has a unique dense element and hence is a Boolean algebra.

For any filter $F$ of a distributive lattice, a congruence relation $\Psi_F$ is defined by $(x, y) \in \Psi_F$ if there exist $f \in F$ such that $x \land f = y \land f$. The associated quotient lattice is denoted by $L_{/\Psi(F)}$ and $\Psi$ denotes the canonical epimorphism of $L$ onto the quotient lattice. For $x \in L$, $\Psi(x) = \hat{x} = \text{the}$
Boolean filters of distributive lattices

congruence class of $x$ modulo $\Psi_F$. It is well-known that the elements of $F$ are all congruent under $\Psi_F$ and the equivalence class of $F$ is the largest element in $L/\Psi_F$. It is also clear that $L/\Psi_F$ is a distributive lattice. This congruence was studied in detail by T.P. Speed [6]. Now, Boolean filters are characterized in terms of congruence $\Psi_F$.

**Theorem 2.12.** Let $F$ be a filter of a pseudo-complemented distributive lattice $L$. Then the following conditions are equivalent.

(1) $F$ is a Boolean filter.

(2) $L/\Psi_F$ is a Boolean algebra.

**Proof.** (1) $\Rightarrow$ (2) : Assume that $F$ is a Boolean filter of $L$. Let $\widehat{x} \in L/\Psi_F$. We have always $x \land x^* = 0$ and hence $\widehat{x} \land \widehat{x}^* = \widehat{x \land x^*} = \widehat{0}$. Since $F$ is a Boolean filter, we get that $x \lor x^* \in F$. Hence we have $\widehat{x} \lor \widehat{x}^* = \widehat{x \lor x^*} = F$. Therefore, $L/\Psi_F$ is a Boolean algebra.

(2) $\Rightarrow$ (1) : Assume that $L/\Psi_F$ is a Boolean algebra. Let $x \in L$. Then $\widehat{x} \in L/\Psi_F$. Since $L/\Psi_F$ is a Boolean algebra, there exists $y \in L$ such that $\widehat{x \land y} = \widehat{x} \land \widehat{y} = \widehat{0}$ and $\widehat{x \lor y} = \widehat{x} \lor \widehat{y} = F$. Hence it follows that $(x \land y, 0) \in \Psi_F$ and $x \lor y \in F$. Since $(x \land y, 0) \in \Psi_F$, there exists $f \in F$ such that $x \land y \land f = 0$ and thus we get $y \land f \leq x^*$. Therefore, we get the following consequence.

\[
x \lor y \in F \text{ and } f \in F \Rightarrow (x \lor y) \land f \in F
\]

\[
\Rightarrow (x \land f) \lor (y \land f) \in F
\]

\[
\Rightarrow (x \land f) \lor x^* \in F \quad \text{since } y \land f \leq x^*
\]

\[
\Rightarrow (x \lor x^*) \land (f \lor x^*) \in F
\]

\[
\Rightarrow x \lor x^* \in F
\]

Therefore, $F$ is a Boolean filter of $L$. 

3 Conclusion

We remark that the concepts of prime distributive lattices and very weakly distributive lattices have been introduced and discussed in [7] and [8]. In the literature, J.C. Varlet and B.M. Schein have pointed out around 1972-1975 that a weakly distributive lattice is not necessarily a prime distributive lattice. It is natural to ask whether the prime pseudo-complemented distributivity and the weakly pseudo-complemented distributivity are equivalent condition for a semilattice or a lattice? In our future research, we will try to answer the above question and to characterize the Stone lattices in terms of Boolean filters.

**References**


