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Characterizations of stability for discrete semigroups of bounded linear operators

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Abstract

Let $\mathbb{T} = \{T(n)\}_{n\geq 0}$ be a discrete semigroup of bounded linear operators acting on a Banach space X. We prove that if for each $\mu \in \mathbb{R}$ and every q-periodic sequence f with $f(0) = 0$, the sequence $n \to \sum_{k=0}^{n} e^{i\mu k} T(n-k) f(k)$ is bounded, then the semigroup $\mathbb{T}$ is uniformly exponentially stable.

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1 Introduction

Let X be a real or complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all linear and bounded operators acting on X. We denote by $\|\cdot\|$ the norms of operators and vectors. Denote by $\mathbb{R}_+$ the set of real numbers and by $\mathbb{Z}_+$ the set of all non-negative integers. Let $B(\mathbb{Z}_+, X)$ be the space of X-valued bounded sequences with supremum norm, and $P_0^q(\mathbb{Z}_+, X)$ be the space of q-periodic (with $q \geq 2$) sequences f with $f(0) = 0$. Then clearly $P_0^q(\mathbb{Z}_+, X)$ is a closed subspace of $B(\mathbb{Z}_+, X)$. We denote by $AP_0(\mathbb{Z}_+, X)$ the space of almost periodic sequences f with $f(0) = 0$. For $T \in \mathcal{B}(X)$, $\sigma(T)$ the spectrum of T and the the spectral radius of T is defined as $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. It is also well known that $r(T) := \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$. The resolvent set of T is defined as $\rho(T) := \mathbb{C}\backslash\sigma(T)$, i.e the set of all $\lambda \in \mathbb{C}$ for which $T - \lambda I$ is an invertible operator in $\mathcal{B}(X)$. A well-known result in [1] says that if X is a Banach space of finite dimension and A a linear and bounded operator acting on X then A is stable if and only if for each $\mu \in \mathbb{R}$ and each $b \in X$ the solution of the discrete Cauchy problem

$$x_{n+1} = Ax(n) + e^{i\mu n}b, \;\; x(0) = 0, \;\; n \in \mathbb{Z}_+$$

is bounded, if and only if there exist positive constants N and ν such that $\|A^n\| \leq Ne^{-\nu n}$ for all $n \geq 0$, or equivalently the spectral radius of A is less than one. In [2] the stability of strongly continuous semi groups are characterized by convolutions. This note is the discrete case of [2] for discrete semigroups of bounded linear operators acting on X. We give some results in the frame work of general Banach space and spaces of sequences as defined above.

2 Preliminary results

Recall that T is power bounded if there exists a positive constant M such that $\|T^n\| \leq M$ for all $n \in \mathbb{Z}_+$. We prove some lemmas which are used in the proofs of main results.

Lemma 2.1. [2] Let $T \in \mathcal{B}(X)$. If there exists $M > 0$ such that

$$\sup_{n\in\mathbb{Z}_+} \|I + T + \cdots + T^n\| = M < \infty \tag{2.1}$$

then T is power bounded and $1 \in \rho(T)$.

Proof: The proof is given in [2], but for convenience we prove this. We have the identity

$$T^{n+1} = I + (T - I)(I + T + \cdots + T^n).$$

By using the inequality (2.1) we get that T is power bounded. Next, suppose that $1 \in \sigma(T)$. Then there exists a sequence $(x_m)_{m\in\mathbb{Z}_+}$ with $x_m \in X$, $\|x_m\| = 1$ and $(I - T)x_m \to 0$ as $m \to \infty$, see [[3], Proposition 2.2, p. 64]. Since T is power bounded, $T^k(I - T)x_m \to 0$ as $m \to \infty$ uniformly for $k \in \mathbb{Z}_+$. Let $N \in \mathbb{Z}_+$, $N > 2M$ and $m \in \mathbb{Z}_+$ such that $\|T^k(I - T)x_m\| \leq \frac{1}{2N}$, $k = 0, 1, \ldots N$. Then

$$\begin{aligned}
M &\geq \|\sum_{k=0}^{N} T^k x_m\| = \|x_m + \sum_{k=1}^{N} T^k x_m\| \\
&= \|x_m + \sum_{k=1}^{N}(x_m + \sum_{j=0}^{k-1} T^j(T - I)x_m)\| \\
&= \|(N + 1)x_m + \sum_{k=1}^{N}\sum_{j=0}^{k-1} T^j(T - I)x_m\| \\
&\geq (N + 1) - \frac{N(N + 1)}{4N} > \frac{N}{2} > M
\end{aligned}$$

which is absurd and hence $1 \in \rho(T)$. ∎

Lemma 2.2. [2] Let $U \in \mathcal{B}(X)$ and $\mu \in \mathbb{R}$. If

$$\sup_{n\in\mathbb{Z}_+} \|\sum_{k=0}^{n} e^{i\mu k}U^k\| = M_\mu < \infty. \tag{2.2}$$

Then U is power bounded and $e^{-i\mu} \in \rho(U)$.

Proof: Let $T = e^{i\mu}U$, then by Lemma 2.1 T is power bounded. Since $\|T\| = \|U\|$, U is power bounded. Again by Lemma 2.1 we have $1 \in \rho(T) = \rho(e^{i\mu}U)$, i.e. $e^{i\mu}U - I$ is invertible, from this we get $U - e^{-i\mu}I$ is invertible. Hence $e^{-i\mu} \in \rho(U)$. ∎

Lemma 2.3. Let $U \in \mathcal{B}(X)$. If the inequality (2.2) is true for all $\mu \in \mathbb{R}$, then $r(U) < 1$.

Proof: From Lemma 2.2 we have U is power bounded. So there exists $M > 0$ such that $\|U^n\| \leq M$ for all $n \in \mathbb{Z}_+$. Then clearly $r(U) = \lim_{n\to\infty} \|U^n\|^{\frac{1}{n}} \leq 1$. But $e^{i\mu} \in \rho(U)$ for all $\mu \in R$ and $\sigma(U)$ is compact. Hence $r(U) < 1$. ∎

3 Main results

We recall that a discrete semigroup is a family $\mathbb{T} = \{T(n) : n \in \mathbb{Z}_+\}$ of bounded linear operators on X which satisfying the following conditions

(1) $T(0) = I$, the identity operator on X,

(2) $T(n + m) = T(n)T(m)$ for all $n, m \in \mathbb{Z}_+$.

It is clear that $T(n) = T^n(1)$ for all $n \in \mathbb{Z}_+$, $T(1)$ is called the algebraic generator of the semigroup $\mathbb{T}$.

The growth bound of $\mathbb{T}$ denoted by $\omega_0(\mathbb{T})$ is defined as $\omega_0(\mathbb{T}) = \inf_{n\in\mathbb{Z}_+}\{\omega \in \mathbb{R} :$ there exists $M_\omega \geq 1$ such that $\|T(n)\| \leq M_\omega e^{\omega n}\}$.

The family $\mathbb{T}$ is uniformly exponentially stable if $\omega_0(\mathbb{T})$ is negative, or equivalently, if there exists $M \geq 1$ and $\omega > 0$ such that $\|T(n)\| \leq Me^{-\omega n}$ for all $n \in \mathbb{Z}_+$. The general theory of semigroups can be found in [3], [4] and [5].

Theorem 3.1. Let $\mathbb{T} = \{T(n) : n \in \mathbb{Z}_+\}$ be a discrete semigroup on X and $\mu \in \mathbb{R}$. If

$$\sup_{n\geq 0} \|\sum_{k=0}^{n} e^{i\mu k} T(n-k) f(k)\| < \infty \tag{3.1}$$

for all $f \in P_0^q(\mathbb{Z}_+, X)$ then $T(1)$ is power bounded and $e^{i\mu} \in \rho(T(1))$.

Proof: Let $n = Nq + r$ for some $N \in \mathbb{Z}_+$, where $r \in \{0, 1, \ldots, q-1\}$.
For each $j \in \mathbb{Z}_+$, we consider the set $A_j = \{1 + jq, 2 + jq, \ldots, q - 1 + jq\}$. Let $B_N = \{Nq + 1, Nq + 2, \ldots, Nq + r\}$ if $r \geq 1$ and $C = \{0, q, 2q, \ldots, Nq\}$. Then clearly $\cup_{j=0}^{N-1} A_j \cup B_N \cup C = \{0, 1, 2, \ldots, Nq + r\}$.

For a fixed non-zero $x \in X$, let us consider the sequence defined as
$f(k) = \begin{cases} (k - jq)[(1 + j)q - k]T(k - jq)x, & \text{if } k \in A_j, \\ 0, & \text{if } k \in \{0, q, 2q, \ldots\}. \end{cases}$ Then clearly $f \in P_0^q(\mathbb{Z}_+, X)$.
Now for this sequence we have

$$\begin{aligned}\sum_{k=0}^{n} e^{i\mu k} T(n-k) f(k) &= \sum_{k=0}^{Nq+r} e^{i\mu k} T(Nq+r-k) f(k) \\ &= \sum_{j=0}^{N-1} \sum_{k=1+jq}^{(q-1+jq)} e^{i\mu k} T(Nq+r-k) f(k) + \\ &\quad \sum_{k=Nq+1}^{Nq+r} e^{i\mu k} T(Nq+r-k) f(k)\end{aligned}$$

$$= \quad I_1 + I_2.$$

where

$$\begin{aligned}
I_1 &= \sum_{j=0}^{N-1} \sum_{k=1+jq}^{(q-1+jq)} e^{i\mu k} T(Nq+r-k)(k-jq)[q-(k-jq)]T(k-jq)x \\
&= \sum_{j=0}^{N-1} T(Nq+r-jq) \sum_{k=1+jq}^{(q-1+jq)} e^{i\mu k}(k-jq)[q-(k-jq)]x \\
&= \sum_{j=0}^{N-1} T(Nq+r-jq) e^{i\mu jq} \sum_{\nu=1}^{q-1} e^{i\mu\nu}\nu(q-\nu)x \\
&= \sum_{j=0}^{N-1} e^{-i\mu(Nq+r-jq)} T(Nq+r-jq) e^{i\mu(Nq+r)} \sum_{\nu=1}^{q-1} e^{i\mu\nu}\nu(q-\nu)x \\
&= \sum_{\alpha=r+q}^{r+Nq} e^{-i\mu\alpha} T^{\alpha}(1) e^{i\mu n} \sum_{\nu=1}^{q-1} e^{i\mu\nu}\nu(q-\nu)x \\
&= \sum_{\alpha=r+q}^{n} e^{-i\mu\alpha} T^{\alpha}(1) y
\end{aligned}$$

with $y = e^{i\mu n} \sum\limits_{\nu=1}^{q-1} e^{i\mu\nu}\nu(q-\nu)x$.
and

$$\begin{aligned}
I_2 &= \sum_{\rho=0}^{r-1} e^{i\mu(Nq+r-\rho)} T(\rho) f(Nq+r-\rho)x \\
&= \sum_{\rho=0}^{r-1} e^{i\mu(Nq+r-\rho)} T(\rho) f(r-\rho)x.
\end{aligned}$$

Hence,

$$\sum_{k=0}^{n} e^{i\mu k} T(n-k) f(k) = \sum_{\alpha=r+q}^{n} e^{-i\mu\alpha} T^{\alpha}(1) y + \sum_{\rho=0}^{r-1} e^{i\mu(n-\rho)} T(\rho) f(r-\rho)x.$$

Now by inequality (3.1) we have I_1 is bounded. That is, $\sup_{n\geq 0} \| \sum\limits_{\alpha=r+q}^{n} e^{-i\mu\alpha} T^{\alpha}(1)\| < \infty$. From this we obtain that

$$\sup_{n\geq 0} \| \sum_{\alpha=0}^{n} e^{-i\mu\alpha} T^{\alpha}(1)\| < \infty.$$

By Lemma 2.2 we conclude that $T(1)$ is power bounded and $e^{i\mu} \in \rho(T(1))$. This completes the proof. ∎

The application of this result to the uniform exponential stability of discrete semigroups is presented as a corollary.

Corollary 3.2. Let $\mathbb{T} = \{T(n) : n \in \mathbb{Z}_+\}$ be a discrete semigroup on X. If condition (3.1) holds for all $\mu \in \mathbb{R}$ and every $f \in P_0^q(\mathbb{Z}_+, X)$, then $r(T(1)) < 1$ and $\mathbb{T}$ is uniformly exponentially stable.

Proof: By Theorem 3.1 we have T is power bounded and $e^{i\mu} \in \rho(U)$ for all $\mu \in \mathbb{R}$. Now by Lemma 2.3 we get that $r(T(1)) < 1$ but $r(T(1)) = e^{\omega_0(\mathbb{T})}$. From this we obtain that $\omega_0(\mathbb{T})$ is negative and hence $\mathbb{T}$ is uniformly exponentially stable. ■

Corollary 3.3. Let $\mathbb{T} = \{T(n) : n \in \mathbb{Z}_+\}$ be a discrete semigroup on X. If condition 3.1 holds for all $\mu \in \mathbb{R}$ and every $f \in AP_0(\mathbb{Z}_+, X)$, then $r(T(1)) < 1$ and $\mathbb{T}$ is uniformly exponentially stable.

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