Point set domination with reference to degree

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Abstract
E. Sampathkumar et al introduced [7] the concept of point set domination number of a graph. A set $D \subseteq V(G)$ is said to be a point set dominating set (psd set), if for every $S \subseteq V - D$ there exists a vertex $u \in D$ such that the subgraph $\langle S \cup \{u\} \rangle$ induced by $S \cup \{u\}$ is connected. The minimum cardinality of a psd set is called the point set domination number of $G$ and is denoted by $\gamma_p(G)$. In this paper psd sets are analysed with respect to the strong [9] domination parameter for separable graphs. The characterization of separable graphs with equal psd number and spsd number is derived.

Key words: separable graph, point set domination, strong point set domination

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1 Introduction
A set $D \subseteq V(G)$ is said to be a strong point set dominating set (spsd set), if for every $S \subseteq V - D$ there exists a vertex $u \in D$ such that the subgraph $\langle S \cup \{u\} \rangle$ induced by $S \cup \{u\}$ is connected and $d(u) \geq d(s)$ for all $s \in S$ where $d(u)$ denote the degree of the vertex $u$. The minimum cardinality of an spsd set is called the strong point set domination number of $G$ and is denoted by $\gamma_{sp}(G)$. A connected graph with at least one cut vertex is called a separable graph. If $B$ is a block of a separable graph $G$ with psd set $B'$, then $(V - B) \cup B'$ is a psd set of $G$ but need not be an spsd set of $G$ as seen in the following discussion. Hence the spsd sets of $G$ are characterized first and then analysed with reference to the spsd sets of the blocks of $G$. The characterization of separable graphs with equal psd number and spsd number is derived. In the following discussion, a graph $G$ always means a connected graph.
Main Results

Definition 2.1. A set $D \subseteq V(G)$ is said to be a strong point set dominating set (spsd set) of $G$ if for every $S \subseteq V - D$ there exists a vertex $u \in D$ such that the subgraph $\langle S \cup \{u\} \rangle$ induced by $S \cup \{u\}$ is connected and $d(u) \geq d(s)$ for all $s \in S$ where $d(u)$ denote the degree of the vertex $u$.

The minimum cardinality of an spsd set is called the strong point set domination number of $G$ and is denoted by $\gamma_{sp}(G)$.

Proposition 2.2. A subset $D$ of $V$ is an spsd set if and only if for every independent set $S \subseteq V - D$ there exists $u \in D$ such that $S \subseteq N(u)$ and $d(u) \geq d(s)$ for all $s \in S$.

Proof. If $D$ is an spsd set of $G$, then the condition follows from the definition of $D$. Conversely, suppose the given condition is satisfied. Let $S \subseteq V - D$ be any set. If $S$ is independent, then by the given condition there exists $u \in D$ such that $\langle S \cup \{u\} \rangle$ is connected and $d(u) \geq d(s)$ for all $s \in S$. If $S$ is not independent, then let $S = S_1 \cup S_2$ where $S_1$ is a maximal independent subset of $S$. Let $s' \in S$ be such that $d(s') = \text{Max}_{s \in S} \{d(s)\}$.

Case (i) $s' \in S_1$.

$S_1$ is a maximal independent subset of $S$ implies there exists $u \in D$ such that $S_1 \subseteq N(u)$ and $d(u) \geq d(s)$ for all $s \in S_1$. Therefore, $d(u) \geq d(s')$. $S_1$ is maximal independent subset of $S$ implies every vertex of $S_2$ is adjacent to at least one vertex in $S_1$. Hence $\langle S_1 \cup S_2 \cup \{u\} \rangle$ is connected. Also $d(u) \geq d(s')$ implies $d(u) \geq d(s') \geq d(s)$ for all $s \in S$. Hence $\langle S \cup \{u\} \rangle$ is connected and $d(u) \geq d(s)$ for all $s \in S$.

Case (ii) $s' \in S_2$.

$s' \in S_2$ implies that $s'$ is adjacent to at least one vertex in $S_1$.

(ii) - (a): $s'$ is adjacent to all vertices in $S_1$.

Every vertex of $S_2$ is adjacent to at least one vertex of $S_1$. Therefore, $\langle S_1 \cup S_2 \rangle$ is connected. Also $s' \in V - D$ implies there exists $u \in D$ such that $us' \in E(G)$ and $d(u) \geq d(s')$. Therefore, $\langle S \cup \{u\} \rangle$ is connected and $d(u) \geq d(s') \geq d(s)$ for all $s \in S$.

(ii) - (b): There are vertices in $S_1$ which are not adjacent to $s'$.

Let $A = \{s \in S_1 / s \notin N(s')\} \cup \{s'\}$.
Then $A$ is independent and therefore there exists $u \in D$ such that $\langle A \cup \{u\} \rangle$ is connected and $d(u) \geq d(a)$ for all $a \in A$. By the definition of $A$, $s' \in A$. Therefore, $\langle S \cup \{u\} \rangle$ is connected and $d(u) \geq d(s') \geq d(s)$ for all $s \in S$. Hence $D$ is an spsd set of $G$.

**Remark 2.3.** In the remaining discussion of this paper, a graph $G$ always means a separable graph.

**Observation 2.4.** If $B$ is a block with spsd set $B'$, then $(V - B) \cup B'$ need not be an spsd set of $G$.

**Proof.** Consider the following figure:

![Figure 1](image)

$B' = \{3, 5\}$ is an spsd set of $B$. Then $(V - B) \cup B' = \{1, 3, 5\}$ is a psd set of $G$ but not an spsd set of $G$ since $d(2) > d(1), d(3), d(5)$.

**Remark 2.5.** If a block $B$ has an spsd set $B'$ containing all cut vertices belonging to $B$, then $(V - B) \cup B'$ is an spsd set of $G$.

**Proof.** Let $S \subseteq V - [(V - B) \cup B']$ be independent. Then $S \subseteq B - B'$. $B'$ is an spsd set of $B$ implies there exists $u \in B'$ such that $S \subseteq N_B(u)$ and $d_B(u) \geq d_B(s)$ for all $s \in S$. Since $B'$ contain all cut vertices belonging to $B$, $d_B(s) = d_G(s)$ for all $s \in S$. Hence $d_G(u) \geq d_B(u) \geq d_B(s) = d(s)$ for all $s \in S$. That is, $S \subseteq N(u)$ and $d_G(u) \geq d_G(s)$ for all $s \in S$. Therefore, $(V - B) \cup B'$ is an spsd set of $G$.

Therefore, separable graphs in which every block has a $\gamma_{sp}$ set containing all cut vertices belonging to $B$ are considered in the following discussion.

**Definition 2.6.** $k_{sp} = \max_{B \in B_G} \{|B| - \gamma_{sp}(B)\}$ where $B_G$ denote the set of all blocks of $G$. 


Remark 2.7.
(i) $\gamma_{sp}(G) \leq n - k_{sp}$.
(ii) $\gamma_{sp}(G) \leq n - \Delta$.

Proof.
(i) : $(V - B) \cup B'$ is an spsd set of $G$ implies $\gamma_{sp}(G) \leq n - (|B - B'|)$. Choose a block $B$ for which $|B - B'| = k_{sp}$. Hence $\gamma_{sp}(G) \leq n - k_{sp}$.
(ii) : $D = V(G) - N(u)$ where $d(u) = \Delta$ is an spsd set of $G$ and hence $\gamma_{sp}(G) \leq n - \Delta$.

Remark 2.8. If $D$ is an $\gamma_{sp}$ set of a separable graph $G$, then there are three cases:
(i) $V - D$ contain vertices of different blocks.
(ii) $V - D \subset B$.
(iii) $V - D = B$ for some block $B$.

Definition 2.9. When $V - D \subset B$, define
$P(B, D) = \{u \in V - D / N(u) \cap (B \cap D) = \phi\}$.

Remark 2.10. If $P(B, D) \neq \phi$, then $\gamma_{sp}(G) = n - \Delta$.

Remark 2.11. $B \cap D$ is an spsd set of $B - P(B, D)$.

Proof. Let $S \subseteq B - P(B, D) - B \cap D$ be an independent subset. Then $S \subseteq V - D$. Therefore, there exists $u \in D$ such that $T \subseteq N(u)$ and $d(u) \geq d(s)$ for all $s \in S$. $S$ is an independent subset implies $u$ is adjacent to more than one vertex in $B$ and hence $u \in B \cap D$.

Case (i) $u$ is not a cut vertex.
Then $d_G(u) = d_B(u)$. Hence $d_B(u) = d_G(u) \geq d_G(s) \geq d_B(s)$ for all $s \in S$. That is, there exists $u \in B \cap D$ such that $S \subseteq N(u)$ and $d_B(u) \geq d_B(s)$ for all $s \in S$. That is, $B \cap D$ is an spsd set of $B - P(B, D)$.

Case (ii) $u$ is a cut vertex.
Then every path connecting a point of $V - D$ to a point of $D - B \cap D$ must contain $u$. Hence $N(s) \cap (D - B \cap D) = \phi$ for all $s \in S$. Therefore, $d_G(s) = d_B(s)$ for all $s \in S$. If there exists no $x \in B \cap D$ such that $S \subseteq N(x)$ with $d_B(x) \geq d_B(s)$ for all
\[ s \in S, \text{then as } N(s) \cap (D - B \cap D) = \phi \text{ there exists no } x \in D \text{ such that } S \subseteq N(x) \text{ and } d_G(x) \geq d_B(x) \geq d_B(s) = d_G(s) \text{ for all } s \in S \text{ which is a contradiction to the fact that } D \text{ is an spsd set of } G. \]

Hence there exists \( x \in B \cap D \) such that \( S \subseteq N(x) \) and \( d_B(x) \geq d_B(s) \) for all \( s \in S \). That is, \( B \cap D \) is an spsd set of \( B - P(B, D) \).

\[ \square \]

**Remark 2.12.** If \( P(B, D) = \phi \), then \( B \cap D \) is an spsd set of \( B \).

**Remark 2.13.** If \( P(B, D) = \phi \), then \( \gamma_{sp}(G) = n - k_{sp} \).

**Proof.** \( P(B, D) = \phi \) implies \( B \cap D \) is an spsd set of \( B \) and hence \( \gamma_{sp}(B) \leq |B \cap D| \). Also \( \gamma_{sp}(B) \geq |B \cap D| \). For, if \( \gamma_{sp}(B) > |B \cap D| \), then \( (V - B) \cup B' \) is an spsd set of \( G \) where \( |B'| = \gamma_{sp}(B) \). Then \( |D| = |(V - B) \cup (B \cap D)| > |(V - B) \cup B'| \). That is, there exists an spsd set \( (V - B) \cup B' \) of \( G \) with cardinality less than \( |D| \) which is a contradiction. Hence \( \gamma_{sp}(B) \geq |B \cap D| \).

Therefore, \( \gamma_{sp}(B) = |B \cap D| \). Hence, \( \gamma_{sp}(G) = |D| = |(V - B) \cup (B \cap D)| = |(V - B) \cup B'| \geq n - k_{sp} \). Therefore, \( \gamma_{sp}(G) = n - k_{sp} \).

\[ \square \]

**Remark 2.14.** If \( V - D = B \) for some block \( B \), then \( \gamma_{sp}(G) = n - \Delta \).

**Proof.** \( V - D = B \) implies \( (V - D) \) is complete. Therefore, \( d(u) \geq |V - D| \) and hence \( |D| \geq n - d(u) \geq n - \Delta \) for any vertex \( u \in V - D \). Therefore, \( \gamma_{sp}(G) = n - \Delta \).

\[ \square \]

**Theorem 2.15.** If \( G \) is a connected graph with cut vertices, then
\( \gamma_{sp}(G) = \min \{ n - \Delta, n - k_{sp} \} \)

**Proof.** Let \( D \) be a minimum spsd set of \( G \). Then \( |D| = \gamma_{sp}(G) \).

**Case (i) \( V - D \) contain vertices of different blocks.**

Then \( V - D \subseteq N(w) \). \( d(w) \geq |V - D| \) implies \( |D| \geq n - d(w) \geq n - \Delta \).

Hence \( |D| \geq n - \Delta \). Therefore, \( |D| = n - \Delta \). That is, \( \gamma_{sp}(G) = n - \Delta \). Hence \( n - \Delta = \gamma_{sp}(G) \leq n - k_{sp} \). That is, \( \gamma_{sp}(G) = \min \{ n - \Delta, n - k_{sp} \} \).

**Case (ii) \( V - D \subseteq B \) for some block \( B \).**

Then if \( P(B, D) \neq \phi \), then \( \gamma_{sp}(G) = n - \Delta \). Hence \( n - \Delta = \gamma_{sp}(G) \leq n - k_{sp} \).

That is, \( \gamma_{sp}(G) = \min \{ n - \Delta, n - k_{sp} \} \).
If $P(B, D) = \emptyset$, then $\gamma_{sp}(G) = n - k_{sp}$.

Hence $n - k_{sp} = \gamma_{sp}(G) \leq n - \Delta$. That is, $\gamma_{sp}(G) = \text{Min} \{n - \Delta, n - k_{sp}\}$.

**Case (iii)** $V - D = B$ for some block $B$.

Then $\gamma_{sp}(G) = n - \Delta$. Hence $n - \Delta = \gamma_{sp}(G) \leq n - k_{sp}$. That is, $\gamma_{sp}(G) = \text{Min} \{n - \Delta, n - k_{sp}\}$. Hence in all cases $\gamma_{sp}(G) = \text{Min} \{n - \Delta, n - k_{sp}\}$. 

\[ \text{Theorem 2.16.} \quad k = k_{sp} \text{ if and only if there exists a block } B \text{ such that } k = |B| - \gamma_p(B) \text{ and } \gamma_p(B) = \gamma_{sp}(B). \]

**Proof.** Let $k = k_{sp}$.

If $B$ is a block with $k_{sp} = |B| - \gamma_{sp}(B)$, then $\gamma_p(B) = \gamma_{sp}(B)$. For, if $\gamma_p(B) \neq \gamma_{sp}(B)$, then $\gamma_p(B) < \gamma_{sp}(B)$. Therefore, $k_{sp} = |B| - \gamma_{sp}(B) < |B| - \gamma_p(B) \leq k$.

This implies $k_{sp} < k$ which is a contradiction $k = k_{sp})$. Hence for any block $B$ for which $k_{sp} = |B| - \gamma_{sp}(B)$, $\gamma_p(B) = \gamma_{sp}(B)$. If $k = k_{sp}$, then $k = |B| - \gamma_{sp}(B) = |B| - \gamma_p(B)$. Hence there exists a block for which $k = |B| - \gamma_p(B)$ and $\gamma_p(B) = \gamma_{sp}(B)$.

Conversely, let there exists a block $B$ such that $k = |B| - \gamma_p(B)$ and $\gamma_p(B) = \gamma_{sp}(B)$, $k = |B| - \gamma_p(B) = |B| - \gamma_{sp}(B) \leq k_{sp}$. Hence $k \leq k_{sp}, \ldots, (1)$.

For any block $B$, $\gamma_p(B) \leq \gamma_{sp}(B)$.

This implies $k = (|B| - \gamma_p(B)) \geq |B| - \gamma_{sp}(B)$. Choose a block $B$ for which $k_{sp} = |B| - \gamma_{sp}(B)$. Then $k = |B| - \gamma_p(B) = k_{sp}, \ldots, (2)$. (1) and (2) together give $k = k_{sp}$.

**Notation 2.17.**

(i) $D_{sp}(G)$ denotes the set of all spsd sets of $G$.

(ii) $D_{sp}(G; X_1)$ denotes the set of all spsd sets $D$ of $G$ with $V - D \subseteq B$ and $P(B, D) = \emptyset$ for some $B \in B_G$.

(iii) $D_{sp}(G; X_1)$ denotes the set of all spsd sets $D$ of $G$ with $V - D \subseteq B$ and $P(B, D) \neq \emptyset$ for some $B \in B_G$.

(iv) $D_{sp}(G; X_1)$ denotes the set of all spsd sets $D$ of $G$ with $V - D = B$.

**Theorem 2.18.** For any separable graph $D_{sp}(G; X_1) \neq \emptyset$. 

Proof. For every block $B$ there exists a $\gamma_{sp}$ set $B'$ containing all cut vertices belonging to $B$. Let $D = (V - B) \cup B'$. Then $V - D = B - B' \subset B$ and $B \cap D = B'$. $D \in D_{sp}(G)$ and $B - B'$ has no cut vertices. Therefore, $N(u) \cap (D - (B \cap D)) = N(u) \cap (V - B) = \phi$ for all $u \in B - B'$. Hence $N(u) \cap (D - B') = \phi$ and $N(u) \cap (B \cap D) = N(u) \cap B' \neq \phi$ for all $u \in V - D$. Therefore, $P(B, D) = \phi$. Hence $V - D \subset B$ with $P(B, D) = \phi$. That is, $D \in D_{sp}(G; X_1)$. Therefore, $D_{sp}(G; X_1) \neq \phi$.

Theorem 2.19. $D_{sp}(G; X_2) \neq \phi$ if and only if there exists $B \in B_G$ such that $B$ can be partitioned into three non empty sets $V_1$, $V_2$ and $V_3$ satisfying the following conditions:

(a) $\langle V_1 \rangle$ is complete, $N(x) \cap V_2 = V_2$, $N(x) \cap V_3 = \phi$ and there exists $u \in N(x) \cap (V - B)$ with $d_G(u) \geq d_G(x)$, for each $x \in V_1$.

(b) $V_1 \cup V_2 \cup V_3 = B$.

(c) $d_B(v) = d_G(v)$ for all $v \in V_2$.

(d) $V_3 \in D_{sp}(V_2 \cup V_3)$.

Proof. (a): Let $D \in D_{sp}(G; X_2)$.

Then there exists $B \in B_G$ such that $V - D \subset B$ and $P(B, D) \neq \phi$. Therefore, $(V - D) - P(B, D) = \phi$ and $B \cap D \neq \phi$. Now, let $V_1 = P(B, D)$. Then $\langle V_1 \rangle$ is complete. Let $V_2 = (V - D) - P(B, D)$ and $V_3 = B \cap D$. Then for each $x \in V_1$, $N(x) \cap V_2 = V_2$, $N(x) \cap V_3 = \phi$ and there exists $u \in D - (B \cap D) = (V - B)$ such that $ux \in E(G)$ and $d_G(u) \geq d_G(x)$.

(b): $V_1 \cup V_2 \cup V_3 = P(B, D) \cup [(V - D) - P(B, D)] \cup (B \cap D) = B$.

(c): $P(B, D) \neq \phi$ implies there exists $u \in P(B, D)$. Then $N(u) \cap (B \cap D) = \phi$ and $N(u) \cap (D - (B \cap D)) \neq \phi$. That is, $u$ is a cut vertex.

Hence every path connecting a point of $B$ and a point of $D - (B \cap D)$ must contain $u$. Therefore, for every $v \in (V - D) - P(B, D) = V_2$, $N(v) \cap (D - B \cap D) = \phi$. That is, $v$ is not a cut vertex. Hence $d_B(v) = d_G(v)$ for all $v \in V_2$.

(d): Let $S \subset (V_2 \cup V_3) - V_3$ be independent. Then $S \subset (V_2 \cup V_3) - V_3 = V_2 = V - P(B, D) - D = (B - B \cap D) - P(B, D) \subset B - P(B, D)$. $B \cap D \in D_{sp}(B - P(B, D))$ implies there exists $u \in B \cap D$ such that $\langle S \cup \{u\} \rangle$ is connected and $d_B(u) \geq d_B(s)$ for all $s \in S$. $s \in S \subset V_2$ implies $d_B(s) = d_G(s)$ for all $s \in S$.  


(by (c)). Hence \( S \cup \{u\} \) is connected and \( d_G(u) \geq d_B(u) \geq d_B(s) = d_G(s) \) for all \( s \in S \).

Conversely, suppose there exists \( B \in B_G \) satisfying (a), (b), (c) and (d). Then, let \( D = V - V_1 \cup V_2 = (V - B) \cup V_3 \).
\[ V - D = B \cap (V - V_3) = V_1 \cup V_2 \subset B. \]
\[ P(B, D) = \{ u \in V - D / N(u) \cap (B \cap D) = \phi \}. \]

By condition (a) \( P(B, D) \neq \phi \) and \( V_1 \subseteq P(B, D) \).

By condition (b) \( P(B, D) \cap V_2 = \phi \). \( P(B, D) \subseteq V - D = V_1 \cup V_2 \).
\[ P(B, D) = V_1. \]

**Claim:** \( D \in D_{sp}(G) \).

Let \( W \subset V - D \) be independent. If \( W \cap P(B, D) \neq \phi \) and \( W = \{ w \} \), then \( w \in P(B, D) \). By condition (a) there exists \( u \in N(w) \cap (V - B) \) with \( d_G(u) \geq d_G(w) \). That is, there exists \( u \in D \) such that \( w \in N(u) \) and \( d_G(u) \geq d_G(w) \). If \( W \cap P(B, D) = \phi \), then \( W \subset V_2 = (V - D) - V_1, V_3 \in D_{sp}(V_2 \cup V_3) \). Therefore, there exists \( u \in V_3 \) such that \( W \subseteq N(u) \) and \( d_B(u) \geq d_B(v) \) for all \( v \in W \).
\[ W \subset V_2 \text{ implies } N(v) \cap (V - B) = \phi \text{ for all } v \in W. \]
Therefore, \( d_G(v) = d_B(v) \) by condition (d) and hence \( d_G(u) \geq d_B(u) \geq d_B(v) = d_G(v) \). Hence \( D \in D_{sp}(G) \) with \( V - D \subset B \) and \( P(B, D) \neq \phi \). This implies \( D \in D_{sp}(G; X_2) \).

**Observation 2.20.** The partition of \( B \) in the above theorem is unique.

**Proof.** For if, there exists another partition \( B_1, B_2, B_3 \) such that

(a) \( \langle B_1 \rangle \) is complete, for each \( x \in B_1 \), \( N(x) \cap B_2 = B_2, N(x) \cap B_3 = \phi \) and there exists \( u \in V - B \) such that \( ux \in E(G) \) and \( d_G(u) \geq d_G(x) \).

(b) \( B_1 \cup B_2 \cup B_3 = B \).

(c) \( d_B(v) = d_G(v) \) for all \( v \in B_2 \).

(d) \( B_3 \in D_{sp}(B_2 \cup B_3) \). \( D = V_3 \cup (V - B) = (V - B) \cup B_3 \). Therefore, \( V_3 = B_3 \). If there exists \( u \in V_1 \) such that \( u \notin B_1 \), then there exists \( d \in B_3 \) such that \( ud \in E(G) \). Hence \( N(u) \cap B_3 \neq \phi \) which implies \( N(x) \cap V_3 \neq \phi \) which is a contradiction. Hence \( V_1 \subseteq B_1 \). If there exists \( u \in B_1 \) such that \( u \notin V_1 \), then there exists \( d \in V_3 \) such that \( ud \in E(G) \). Hence \( N(u) \cap V_3 \neq \phi \) which implies \( N(u) \cap B_3 \neq \phi \) with \( u \in B_1 \) which is a contradiction. Hence \( B_1 = V_1 \). Therefore, \( B_2 = V_2 \). That is, the partition is unique.
Theorem 2.21. $D_{sp}(G; Y) \neq \emptyset$ if and only if there exists $B \in B_G$ such that the following conditions are satisfied:

(a) $\langle B \rangle$ is complete.

(b) For each $x \in B$, there exists $u \in N(x) \cap (V - B)$ with $d_G(u) \geq |B|$.

Proof. Let $D \in D_{sp}(G; Y)$. Then there exists $B \in B_G$ with $V - D = B$. That is, $B \cap D = \emptyset$. Hence $N(x) \cap (B \cap D) = \emptyset$ for every $x \in V - D$. $D \in D_{sp}(G)$ implies for every $x \in V - D(= B)$ there exists $u \in N(x) \cap D$. Therefore, $P(B, D) = V - D = B$ and hence $\langle B \rangle = \langle P(B, D) \rangle$ is complete. Hence $d_G(x) \geq |B|$.

Conversely, suppose the above two conditions are satisfied. Let $D = V - B$. Then $V - D = B = P(B, D)$. Therefore, $D \in D_{sp}(G; Y)$. $\square$

Observation 2.22. $D_{sp}(G; Z) \neq \emptyset$ if $G$ has a cut vertex $w$ with $d(w) \geq d(v)$ for all $v \in N(w)$.

Proof. If $D = V - N(w)$, then $V - D = N(w)$ contain vertices of different blocks, and $D \in D_{sp}(G)$. Hence $D \in D_{sp}(G; Z)$. $\square$

Observation 2.23. If $\Delta > k_{sp}$, then there exists a vertex $u$ of degree $\Delta$ such that $N(u)$ is not contained in a single block.

Proof. If for every vertex $u$ of degree $\Delta$ there exists a block $B$ such that $N(u) \subseteq B$, then $\Delta > k_{sp} > 1$ implies $N[u] \subseteq B$. Therefore, $u \in B$. Hence $B - N(u) \in D_{sp}(B)$. But then $\gamma_{sp}(B) \leq |B| - |N(u)|$. That is, $|N(u)| \leq |B| - \gamma_{sp}(B) \leq k_{sp}$. That is, $\Delta \leq k_{sp}$ which is a contradiction. Hence if $\Delta > k_{sp}$ there exists a vertex $u$ of degree $\Delta$ such that $N(u)$ is not contained in a single block. $\square$

Notation 2.24.

(i) $D_{sp}^o(G)$ - denote the set of all minimum spsd sets of $G$.

(ii) $D_{sp}^o(G; X_1) = D_{sp}^o(G) \cap D_{sp}(G; X_1)$.

(iii) $D_{sp}^o(G; X_2) = D_{sp}^o(G) \cap D_{sp}(G; X_2)$.

(iv) $D_{sp}^o(G; Y) = D_{sp}^o(G) \cap D_{sp}(G; Y)$.

(v) $D_{sp}^o(G; X_1) = D_{sp}^o(G) \cap D_{sp}(G; Z)$. 
Remark 2.25. The following theorems are the immediate consequences of the previous results.

Theorem 2.26. $D_{sp}^a(G; Z) = \phi$ if and only if one of the following two conditions is satisfied.

(i) $\Delta < k_{sp}$.

(ii) $\Delta = k_{sp}$ and for every vertex $u$ of degree $\Delta$, $N(u) \subset B$ for some $B \in B_G$.

Theorem 2.27. $D_{sp}^a(G; Z) \neq \phi$ if and only if one of the following two conditions is satisfied.

(i) $\Delta > k_{sp}$.

(ii) $\Delta = k_{sp}$ and there exists a vertex $u$ of degree $\Delta$ such that $N(u)$ is not contained in a single block.

Theorem 2.28. $D_{sp}^a(G; X_1) \neq \phi$ if and only if $\Delta \leq k_{sp}$.

Definition 2.29. If $A \subset V(G)$, then $N(A)$ is the set of all neighbours of vertices in $A$ and $N[A] = A \cup N(A)$.

Definition 2.30. For a complete block $B$, $B^{+\Delta}$ is obtained by the adjunction of one vertex each at every vertex of the block such that degrees of the adjoined vertices in the resulting graph are $\Delta$.

Example 2.31.

![Diagram of a graph with vertices labeled 1 to 20 and a block labeled B.](image)

$B^{+\Delta} = \langle \{1, 2, 3, 4, 5, 6, 7, 8\} \rangle$.

$D = \{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20\}$. 

Figure 2.
Point set domination with reference to degree

\[ V - D = \{1, 2, 3, 4\}. \]
\[ V - D = B, B \cap D = \phi, V - D = P(B, D). \]

**Theorem 2.32.** \( D^p_{sp} (G; Y) \neq \phi \) if and only if

(i) \( \Delta \geq k_{sp} \)

(ii) \( G \) has a block \( B \) which is a clique of order \( \Delta \) and \( \langle |N(B)| \rangle = B^+ \Delta. \)

**Theorem 2.33.** \( D^p_{sp} (G; Y) \neq \phi \) implies \( \Delta \leq k_{sp} + 1. \)

**Remark 2.34.** If \( D^p_{sp} (G; Y) \neq \phi \), then \( k_{sp} \leq \Delta \leq k_{sp} + 1. \)

**Theorem 2.35.** \( D^p_{sp} (G; X_2) \neq \phi \) if and only if the following condition are satisfied.

(i) \( \Delta \geq k_{sp}. \)

(ii) \( V \) can be partitioned into four non empty sets \( V_1, V_2, V_3 \) and \( V_4 \) such that

(a) \( V_1 \neq \phi. \)

(b) \( |V_1 \cup V_2| = \Delta. \)

(c) \( V_1 \cup V_2 \cup V_3 = B \) for some \( B \in B_G. \)

(d) \( V_3 \in D_{sp}(V_2 \cup V_3). \)

(e) \( \langle V_4 \rangle \) is complete, for every \( x \in V_1, N(x) \cap V_2 = V_2, N(x) \cap V_3 = \phi \) and there exists \( u \in N(x) \cap V_4 \) with \( d(u) = \Delta. \)

**Theorem 2.36.** If \( \Delta > k_{sp}, \) then the following statements are valid.

(a) \( N(u) \) is not contained in \( B \) for any \( B \in B_G \) for any vertex \( u \) of degree \( \Delta. \)

(b) \( V - N(u) \in D^0_{sp}(G; Z) \) for any vertex \( u \) of degree \( \Delta. \)

(c) \( |N(u) \cap B| \leq \Delta - 1 \) for any \( B \in B_G \) and for any vertex \( u \) of degree \( \Delta. \)

(d) \( D^0_{sp}(G; X_1) = \phi. \)

(e) \( D^0_{sp}(G; Z) \neq \phi. \)

**Theorem 2.37.** \( D^p_{sp} (G; X_2) \neq \phi \) implies \( \Delta \leq k_{sp} + 1. \)

**Observation 2.38.** \( \gamma_{sp}(G) = n - \Delta \) if and only if \( \Delta \geq k_{sp}. \)
**Observation 2.39.** \( \gamma_{sp}(G) = n - k_{sp} \) if and only if \( \Delta \leq k_{sp} \).

**Theorem 2.40.** \( \gamma_p(G) = \gamma_{sp}(G) \) if and only if one of the following three conditions is satisfied.

(i) \( \Delta > k \).

(ii) \( \Delta = k \) and \( G \) has a cut vertex with \( d(u) = \Delta \).

(iii) \( \Delta \leq k \) and \( k = k_{sp} \).

**Proof.** Let \( \gamma_p(G) = \gamma_{sp}(G) \). If \( \gamma_p(G) = n - \Delta \), then \( \gamma_{sp}(G) = n - \Delta \). Then one of the two conditions (i) or (ii) is satisfied. If \( \gamma_p(G) = n - k \), then \( \gamma_{sp}(G) = n - k \).

**Case (i)** \( \gamma_{sp}(G) = n - \Delta \).

\( \gamma_p(G) = n - k = n - \Delta = \gamma_{sp}(G) \), then \( \Delta = k \) and has a cut vertex \( u \) with \( d(u) = \Delta \).

**Case (ii)** \( \gamma_{sp}(G) = n - k_{sp} \).

Then \( \gamma_p(G) = n - k \) and \( \gamma_p(G) = \gamma_{sp}(G) \) implies \( n - k = \gamma_p(G) = n - k_{sp} = \gamma_{sp}(G) \). Therefore, \( k = k_{sp} \) and \( \gamma_p(G) = n - k \) implies \( n - k \leq n - \Delta \). That is, \( k \geq \Delta \).

Conversely, If (i) is satisfied, then \( n - \Delta < n - k \). Hence \( \gamma_p(G) = n - \Delta \).

\( \gamma_p(G) = \gamma_{sp}(G) \leq \gamma_{sp}(G) \). Therefore, \( \gamma_{sp}(G) = n - \Delta \). Therefore, \( \gamma_p(G) = \gamma_{sp}(G) \).

If (ii) is satisfied, then \( \gamma_p(G) = n - \Delta \) and hence \( \gamma_{sp}(G) = n - \Delta \). That is, \( \gamma_p(G) = \gamma_{sp}(G) \).

If (iii) is satisfied, then \( \Delta \geq k \) implies \( n - \Delta \leq n - k \) and hence \( \gamma_p(G) = n - k \).

Therefore, \( \gamma_p(G) = n - k = n - k_{sp} \). Hence \( n - k_{sp} = \gamma_p(G) \leq \gamma_{sp}(G) \). That is, \( \gamma_{sp}(G) = n - k_{sp} \). Therefore, \( \gamma_p(G) = \gamma_{sp}(G) \).

\[
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**References**

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