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On 2-rainbow domination of some families of graphs

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Abstract

In this paper we find the $2-$rainbow domination number of the 4-regular Harary graphs $H_{4,n}$, $n \geq 5$. Upper bound for the 2-rainbow domination number of $P_1 \times P_m$ is found. At the end we find the lower bound for the 2-rainbow domination number of k-regular graphs.

Keywords: Domination, Harary graphs, grids, k-regular graph.

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1 Notations and Preliminary results

A subset S of the vertex set $V(G)$ of a graph G is called a *dominating set* if every vertex in $V(G) \setminus S$ is adjacent to a vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G.
Let G be a graph and $v \in V(G)$. The open neighborhood of v is the set $N(v) = \{u \in V(G) \mid uv \ \in E(G)\}$ and its closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. Let $f : V(G) \rightarrow \wp\{1, 2, ..., k\}$ be a function that assigns to each vertex of G a set of colors chosen from the power set of $\{1, 2, ..., k\}$. If for each vertex $v \in V(G)$ with $f(v) = \phi$,

$$\bigcup_{u \in N(v)} f(u) = \{1, 2, ..., k\},$$

then the function f is called a $k-$rainbow dominating function $(kRDF)$ of G. The weight of the function f, denoted by $w(f)$ is defined as

$$w(f) = \sum_{v \in V(G)} |f(v)|.$$

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The minimum weight of a $kRDF$ is called the k-rainbow domination number of G and is denoted by $\gamma_{rk}(G)$.

In this paper we consider the $2-$rainbow domination, defined as $f : V(G) \rightarrow \wp\{1,2\}$ such that for each vertex $v \in V(G)$ with $f(v) = \phi$, we have

$$\bigcup_{u \in N(v)} f(u) = \{1,2\}.$$

Such a function f is called a $2-$rainbow dominating function $(2RDF)$ and minimum weight of such function is called the $2-$rainbow domination number of G and is denoted by $\gamma_{r2}(G)$. The cartesian product of G_1 and G_2 denoted by $G_1 \times G_2$ is a graph with vertex set $V(G_1 \times G_2) = \{(v_i, u_j) | v_i \in V(G_1) \ \ and \ \ u_j \in V(G_2)\}$ and $(v_i, u_j)(v_l, u_k) \in E(G_1 \times G_2)$, *iff* $v_i = v_l$ and $u_j u_k \in E(G_2)$ or $u_j = u_k$ and $v_i v_l \in E(G_1)$.

Throughout this paper we denote a path on $n+1$ vertices by P_n .
If $1 \leq k < n$, the Harary graph $H_{k,n}$ of order n is constructed as follows:

The n vertices are placed on the circumference of a circle.

(a) If $k = 2r$, join each vertex to the nearest r vertices in each direction around the circle.

(b) If $k = 2r + 1$ and n is even, join each vertex to the nearest r vertices in each direction on the circle and also to the vertex exactly opposite to it.

(c) Suppose $k = 2r + 1$ and n is odd. First the graph $H_{2r,n}$, is constructed as in part (b). Define $tn + i = i$ for any positive integer t and using this (modulo) addition rule, construct additional edges by joining vertex i and vertex $\frac{n+3}{2}$ for $1 \leq i \leq \frac{n+1}{2}$.

Some known results about the $2-$rainbow Domination in graphs are given below.

Theorem 1.1. *[1]* $2-$*rainbow dominating function is NP-complete.*

Theorem 1.2. *[1]* $2-$*rainbow dominating function is NP-complete even when restricted to chordal graphs.*

Theorem 1.3. *[1] $2-$rainbow dominating function is NP-complete even when restricted to bipartite graphs.*

Theorem 1.4. *[1] $\gamma_{r2}(P_n) = \lfloor \frac{n}{2} \rfloor + 1$.*

Theorem 1.5. *[1] For $n \geq 3$, $\gamma_{r2}(C_n) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$.*

Theorem 1.6. *[1] For $n \geq 3$, $\gamma_{r2}(S_n) = n$.*

Theorem 1.7. *[1] For the generalized Petersen graph $GP(n, k)$,*

$$\gamma_{r2}(GP(n, k)) \leq n.$$

Theorem 1.8. *[1] For any relatively prime numbers n and k, with $k < n$,*

$$\gamma_{r2}(GP(n, k)) \geq \lceil \frac{4n}{5} \rceil.$$

Theorem 1.9. *[1] $\gamma_{r2}(GP(5, 2)) = 5$.*

2 On 2-rainbow domination number of 4-regular Harary graph

In this section we find the $2-$rainbow domination number of the 4-regular Harary graphs $H_{4,n}$, $n \geq 5$. Also the upper bound for the 2-rainbow domination number of $P_1 \times P_m$ iand the lower bound for the 2-rainbow domination number of k-regular graphs are found in this paper.

Proposition 2.1. *For $n \geq 5$, $\gamma(H_{4,n}) \leq \lceil \frac{n}{5} \rceil$.*

Proof. In a Harary graph $H_{4,n}$, $n \geq 5$ a vertex say v is adjacent to four other vertices. Thus there exists a set of five vertices among which one vertex is adjacent to the remaining four vertices in Harary graph. We choose that vertex as an element of the dominating set. Thus we can partition the vertex set of $H_{4,n}$, $n \geq 5$ into $\lceil \frac{n}{5} \rceil$ subsets. We form a set S such that $|S| = \lceil \frac{n}{5} \rceil$ and exactly one element of the above $\lceil \frac{n}{5} \rceil$ subsets is an element of S. Thus $\gamma(H_{4,n}) \leq \lceil \frac{n}{5} \rceil$. ∎

Proposition 2.2. *For any connected graph G, $\gamma_{r2}(G) \leq 2\gamma_r(G)$.*

Proof. Let S be a dominating set of G with $|S| = \gamma_r(G)$. Define

$$f : V(G) \to \wp\{1,2\}$$

such that

$$f(x_i) = \begin{cases} \{1,2\}, & \text{if } x_i \in S; \\ \phi, & \text{if } x_i \notin S \quad \text{where } i = 1, ..., n. \end{cases}$$

This function is a $2RDF$ of G and we have $\gamma_{r2}(G) \leq 2\gamma_r(G)$. ∎

Corollary 2.3. *For* $n \geq 5$, $\gamma_{r2}(H_{4,n}) \leq 2\lceil \frac{n}{5} \rceil$.

Proof. Let $H_{4,n}$, $n \geq 5$ be a Harary graph. Let S be the set as defined in proposition 2.1.
Define $f : V(H_{4,n}) \to \wp\{1,2\}$ such that

$$f(x_i) = \begin{cases} \{1,2\}, & \text{if } x_i \in S; \\ \phi, & \text{if } x_i \notin S \quad \text{where } i = 1, ..., n. \end{cases}$$

$$\bigcup_{u_i \in N(x_i)_{x_i \notin S}} f(u_i) = \{1,2\} \quad \text{where } i = 1, ..., n.$$

Clearly, f is a $2-$ rainbow domination function and since each vertex in the dominating set is assigned with a set of two colors and there are a total of $\lceil \frac{n}{5} \rceil$ vertices in the dominating set of $H_{4,n}$, $n \geq 5$, we deduce that $\gamma_{r2}(H_{4,n}) \leq 2\lceil \frac{n}{5} \rceil$, $n \geq 5$. ∎

Theorem 2.4. *For* $n \geq 5$, $\gamma_{r2}(H_{4,n}) \geq \lceil \frac{n}{3} \rceil$.

Proof. Let $H_{4,n}$ be a graph with $|V(H_{4,n})| = n$. Let $f : V(H_{4,n}) \to \wp\{1,2\}$ be a $2RDF$ of $H_{4,n}$ of minimum weight. Let $S = \{x \in V(H_{4,n}) : f(x) \neq \phi\}$. Then for every $u \in V(H_{4,n}) \setminus S$, we have $|f(N(u))| \geq 2$. By summing up these inequalities for all vertices of $V(H_{4,n}) \setminus S$ we get,

$$\sum_{u \in V(H_{4,n}) \setminus S} |f(N(u))| \geq 2(|V(H_{4,n})| - |S|).$$

That is, $\sum_{u \in V(H_{4,n}) \setminus S} |f(N(u))| \geq 2(|V(H_{4,n})| - \gamma_{r2}(H_{4,n}))$ where every vertex of S is adjacent to 4 vertices of $V(H_{4,n}) \setminus S$. So, each weight is counted exactly 4

times on the left hand-side of the above inequality.
Thus,

$$4\gamma_{r2}(H_{4,n}) \geq 2(|V(H_{4,n})| - \gamma_{r2}(H_{4,n})),$$

$$6\gamma_{r2}(H_{4,n}) \geq 2|V(H_{4,n})|,$$

$$6\gamma_{r2}(H_{4,n}) \geq 2n,$$

$$\gamma_{r2}(H_{4,n}) \geq \frac{n}{3}.$$

By definition $\gamma_{r2}(H_{4,n})$ is an integer. Hence $\gamma_{r2}(H_{4,n}) \geq \lceil \frac{n}{3} \rceil$, $n \geq 5$. ■

Proposition 2.5. *For* $P_1 \times P_m$, $\gamma(P_1 \times P_m) \leq \lceil \frac{m+2}{2} \rceil$, *where* $m \in N$.

Proof. The strip $P_1 \times P_m$ is a graph on $2m + 2$ vertices. The elements for a dominating set S can be selected from $P_1 \times P_m$ as given below:
Let us suppose that $(v_1, u_1) \in S$. Then $(v_1, u_j) \in S$ if $j \equiv 1(mod\ 4)$ and $(v_2, u_j) \in S$ if $j \equiv 3(mod\ 4)$ where $j \leq m + 1$.
If $m+1 \equiv 0(mod\ 4)$ then (v_1, u_{m+1}) may be included in S to dominate (v_1, u_{m+1}).
If $m+1 \equiv 2(mod\ 4)$ then (v_2, u_{m+1}) may be included in S to dominate (v_2, u_{m+1}).
If $m + 1 \equiv 1(mod\ 4)$ then (v_1, u_{m+1}) must be in S.
If $m + 1 \equiv 3(mod\ 4)$ then (v_2, u_{m+1}) must be in S.
Thus, S is a dominating set with $2 + \lceil \frac{2m-4}{4} \rceil$ vertices.

Hence $\gamma(P_1 \times P_m) \leq \lceil \frac{m+2}{2} \rceil$. ■

Corollary 2.6. $\gamma_{r2}(P_1 \times P_m) \leq 2\lceil \frac{m+2}{2} \rceil$.

Proof. The $P_1 \times P_m$ is a graph on $2m+2$ vertices. Let S be the set (dominating set) such that $S = \{x \in V(P_1 \times P_m) : f(x) \neq \phi\}$. Then for every $u \in V(P_1 \times P_m) \setminus S$, we have $|f(N(u))| \geq 2$. Let $f : V(P_1 \times P_m) \to \wp\{1, 2\}$ be defined as

$$f(x_i) = \begin{cases} \{1, 2\}, & \text{if } x_i \in S; \\ \phi, & \text{if } x_i \notin S \text{ where } i = 1, 2, ..., n. \end{cases}$$

$$\bigcup_{u_i \in N(x_i)_{x_i \notin S}} f(u_i) = \{1, 2\} \quad \text{where } i = 12, , ..., n.$$

Such a function f is a $2-$ rainbow domination function and since each vertex in the dominating set is assigned with a set of two colors and there are a total of $\lceil \frac{m+2}{2} \rceil$ vertices in dominating set of $P_1 \times P_m$, we deduce that $\gamma_{r2}(P_1 \times P_m) \leq 2\lceil \frac{m+2}{2} \rceil$. ∎

Theorem 2.7. *Let G be a k-regular graph, then $\gamma_{r2}(G) \geq \lceil \frac{2n}{k+2} \rceil$.*

Proof. Let G be a $k-$ regular graph with $|V(G)| = n$ and f be a $2RDF$ of G of minimum weight. Let S be the set (dominating set) such that $S = \{x \in V(G) : f(x) \neq \phi\}$. Then for every $u \in V(G) \setminus S$, we have $|f(N(u))| \geq 2$. By summing up these inequalities for all vertices of $V(G) \setminus S$ we get,

$$\sum_{u \in V(G)\setminus S} |f(N(u))| \geq 2(|V(G)| - |S|),$$

$$\sum_{u \in V(G)\setminus S} |f(N(u))| \geq 2(|V(G)| - \gamma_{r2}(G)).$$

where every vertex of S is adjacent to k vertices (from $V(G)\setminus S$). The left hand-side of the above inequality shows that each weight is counted exactly k times. Thus,

$$k\gamma_{r2}(G) \geq 2(|V(G)| - \gamma_{r2}(G))$$

$$(k+2)\gamma_{r2}(G) \geq 2|V(G)|$$

$$(k+2)\gamma_{r2}(G) \geq 2n$$

$$\gamma_{r2}(G) \geq \frac{2n}{k+2}$$

Since $\gamma_{r2}(G)$ is an integer, $\gamma_{r2}(G) \geq \lceil \frac{2n}{k+2} \rceil$. ∎

Remark 2.8. *Theorem 1.3 and Theorem 1.5 are the corollaries of the above theorem.*

Corollary 2.9. *[1] For $n \geq 5$, $\gamma_{r2}(H_{4,n}) \geq \lceil \frac{n}{3} \rceil$.*

Corollary 2.10. *[1]For any relatively prime numbers n and k, with $k < n$, $\gamma_{r2}(GP(n,k)) \geq \lceil \frac{4n}{5} \rceil$.*

3 Conclusion

We conclude this paper with the following open problems.

Open Problem 3.1. *Calculate $2-$rainbow domination number of $H_{k,n}$ where k is even other than 4 or k is odd and n is even or k and n are both odd.*

Open Problem 3.2. *Compute more sharp lower bound for $2-$rainbow domination number of $k-$regular graph.*

Open Problem 3.3. *Find the upper bound of $2-$rainbow domination number for $k-$regular graph.*

Open Problem 3.4. *Calculate $2-$rainbow domination number for $P_t \times P_m$ where t is other than 1.*

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