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Some new graceful graphs

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Abstract

In this paper we show that the graphs obtained by duplication of an arbitrary vertex in cycle C_n as well as duplication of an arbitrary edge in even cycle C_n are graceful graphs. In addition, we derive that the joint sum of two copies of cycle C_n admits graceful labeling.

Keywords: Graceful graphs, Duplication of a vertex, Joint sum.

AMS Subject Classification(2010): 05C78.

1 Introduction

We begin with simple, finite, connected and undirected graph $G = (V, E)$ with p vertices and q edges. For all other standard terminology and notations we follow Harary[6]. We give a brief summary of definitions and other information which are useful for the present investigations.

Definition 1.1. *If the vertices are assigned values subject to certain conditions then it is known as graph labeling.*

For a dynamic survey on graph labeling we refer to Gallian[4]. A systematic study of various applications of graph labeling is carried out in Bloom and Golomb[2].

Definition 1.2. *A function f is called graceful labeling of a graph G if $f : V \to \{0, 1, \ldots, q\}$ is injective and the induced function $f^* : E \to \{1, \ldots, q\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. A graph which admits graceful labeling is called a graceful graph.*

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Rosa[7] introduced such labeling in 1967 and named it as a $\beta - valuation$ of graph while Golomb[5] independently introduced such labeling and called it as *graceful labeling*. Acharya[1] has constructed certain infinite families of graceful graphs from a given graceful graph while Rosa[7] and Golomb[5] have discussed gracefulness of complete bipartite graphs and Eulerian graphs. Gracefulness of union of two path graphs with grid graphs and complete bipartite graphs are discussed in [9] by Vaidya et al. Sekar[8] has proved that the splitting graph (the graph obtained by duplicating the vertices of a given graph altogether) of C_n admits graceful labeling for $n \equiv 1, 2(mod\ 4)$ while we prove the duplication of an arbitrary vertex in C_n produces a graceful graph for all n. Moreover the gracefulness of joint sum of graceful trees is discussed by Jin et al.[3] while we investigate graceful labeling for the joint sum of two copies of cycle C_n.

Definition 1.3. *Duplication of a vertex v_k of a graph G produces a new graph G_1 by adding a new vertex v'_k in such a way that $N(v_k) = N(v'_k)$.*

Definition 1.4. *Duplication of an edge $v_i v_{i+1}$ of a graph G produces a new graph G_1 by adding a new edge $v'_i v'_{i+1}$ in such a way that $N(v'_i) = N(v_i) \bigcup \{v'_{i+1}\} - \{v_{i+1}\}$ and $N(v'_{i+1}) = N(v_{i+1}) \bigcup \{v'_i\} - \{v_i\}$.*

Definition 1.5. *Consider two copies of C_n, connect a vertex of the first copy to a vertex of second copy with a new edge, the new graph obtained is called the joint sum of C_n.*

2 Main Results

Theorem 2.1. *Duplication of an arbitrary vertex of C_n produces a graceful graph.*

Proof. Let v_1,v_2,..., v_n be the vertices of the cycle C_n and G be the graph obtained by duplicating an arbitrary vertex of C_n. Without loss of generality let this vertex be v_1 and the newly added vertex be v'_1. To define $f : V \to \{0, 1, \ldots, q\}$ following four cases are to be considered.

Case (i) $n \equiv 0(mod\ 4)$ **;** $n \neq 4$

$f(v'_1) = 0$

$f(v_1) = \frac{n}{2} + 1$

For $2 \leq i \leq \frac{n}{2}+2$

$f(v_i) = (n+2) - (\frac{i-2}{2})$; when i is even.

$= \frac{i-1}{2}$; when i is odd.

For $\frac{n}{2}+3 \leq i \leq n$

$f(v_i) = (n+2) - (\frac{i-1}{2})$; when i is odd.

$= \frac{i-2}{2}$; when i is even.

The case when $n = 4$ is to be dealt separately and the graph is labeled as shown in Figure 1.

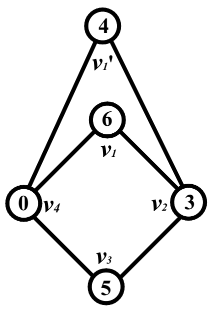

Figure 1: Duplication of a vertex in C_4 and its graceful labeling

Case (ii) $n \equiv 1(mod\,4)$

$f(v'_1) = 0$

$f(v_1) = \frac{n+1}{2}$

For $2 \leq i \leq \frac{n+1}{2}$

$f(v_i) = (n+2) - (\frac{i-2}{2})$; when i is even.

$= \frac{i-1}{2}$; when i is odd.

For $\frac{n+3}{2} \leq i \leq n$

$f(v_i) = \frac{i}{2}$; when i is even.

$= (n+2) - (\frac{i-1}{2})$; when i is odd.

Case (iii) $n \equiv 2(mod\,4)$ **;** $n \neq 6$

$f(v'_1)$=0

$f(v_1) = \frac{n}{2}$+3

For 2≤i≤ $\frac{n+4}{2}$

$f(v_i) = (n+2) - (\frac{i-2}{2})$; when i is even.

$= \frac{i-1}{2}$; when i is odd.

$f(v_i)$=$\frac{i}{2}$; for $i = \frac{n+4}{2}+1$

For $\frac{n+8}{2} \leq i \leq n$

$f(v_i) = (n+2) - (\frac{i-3}{2})$; when i is odd.
$= \frac{i+2}{2}$; when i is even.

For $n = 6$; the corresponding graph and its graceful labeling is shown in Figure 2.

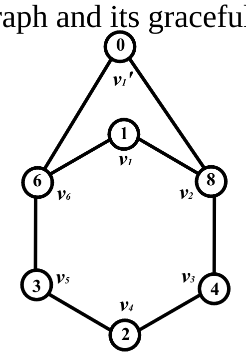

Figure 2: Duplication of a vertex in C_6 and its graceful labeling

Case (iv) $n \equiv 3(mod\, 4)$

$f(v_1')$=0

$f(v_1) = \frac{n+1}{2}$

For 2≤i≤ $\frac{n+1}{2}$

$f(v_i) = (n+2) - (\frac{i-2}{2})$; when i is even.
$= \frac{i-1}{2}$; when i is odd.

For $\frac{n+3}{2} \leq i \leq n$

$f(v_i) = (n+2) - (\frac{i-1}{2})$; when i is odd.
$= \frac{i}{2}$; when i is even.

In view of the above defined labeling pattern f is a graceful labeling for the graph obtained by the duplication of an arbitrary vertex in cycle C_n. ∎

Illustration 2.2. *The graph obtained by duplicating the vertex v_1 of cycle C_9 is shown in Figure 3.*

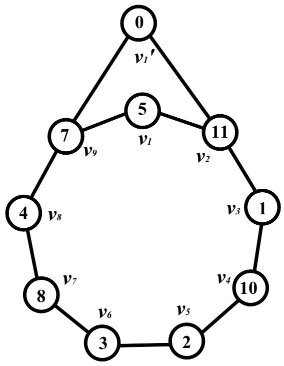

Figure 3: The graceful labeling of duplication of a vertex in C_9

Theorem 2.3. *Duplication of an edge in cycles of even order admits graceful labeling.*

Proof. Let v_1,v_2,..., v_n be the vertices of the cycle C_n where n is even and G be the graph obtained by duplicating an arbitrary edge of C_n. Without loss of generality we may assume that $e' = v_1'v_2'$ be the newly added edge to duplicate the edge $e = v_1v_2$ in C_n. To define $f : V \rightarrow \{0, 1, \ldots, q\}$ following two cases are to be considered.

Case (i) $n \equiv 0(mod\, 4)$**;** $n \neq 4, n \neq 8$

$f(v_1') = \frac{n}{2} + 4$

$f(v_2') = \frac{n}{2}$

For $1 \leq i \leq \frac{n}{2} + 2$

$f(v_i)$ =$(n + 3) - \frac{i-1}{2}$; when i is odd.

=$\frac{i-2}{2}$; when i is even.

$f(v_i)$ =$\frac{i-1}{2}$; for $i = \frac{n}{2} + 3$

For $\frac{n}{2} + 4 \leq i \leq n - 1$

$f(v_i)$ =$(n + 3) - \frac{i}{2}$; when i is even.

=$\frac{i-1}{2}$; when i is odd.

$f(v_n)$ =$\frac{n}{2} + 2$

The labeling for the graphs corresponding to C_4 and C_8 are to be dealt separately. The labeling pattern is provided in Figure 4.

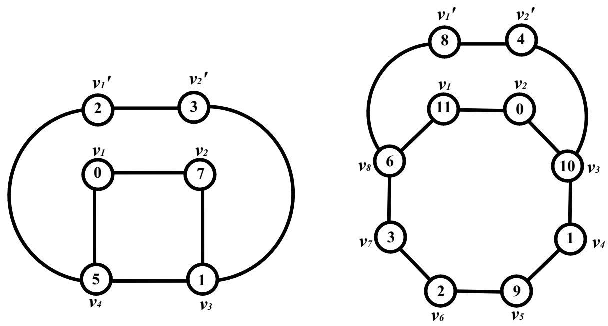

Figure 4: Graceful labeling of edge duplication in C_4 and C_8

Case (ii) $n \equiv 2(mod\, 4)$

$f(v_1') = \frac{n}{2} - 1$

$f(v_2') = \frac{n}{2}$

For $1 \leq i \leq \frac{n}{2}+2$

$f(v_i)$ =$(n+3)-\frac{i-1}{2}$; when i is odd.

=$\frac{i-2}{2}$; when i is even.

For $\frac{n}{2}+3 \leq i \leq n$

$f(v_i)$ =$(n+3)-\frac{i+2}{2}$; when i is even.

=$\frac{i-3}{2}$; when i is odd.

The above defined function f provides graceful labeling for the graph obtained by the duplication of an edge in cycle C_n. ∎

Illustration 2.4. *Consider the graph obtained by duplicating an edge v_1v_2 in C_{10}. The corresponding graceful labeling is shown in Figure 5.*

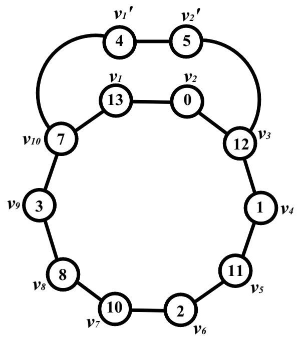

Figure 5: Edge duplication in C_{10} and its graceful labeling

Theorem 2.5. *The joint sum of two copies of cycle C_n admits graceful labeling.*

Proof. We denote the vertices of first copy of C_n by v_1,v_2,....v_n and second copy by v_{n+1}, v_{n+2},v_{n+3},....v_{2n}. Join the two copies of C_n with a new edge and G be the resultant graph. Without loss of generality we assume that the new edge be v_nv_{n+1}, so that v_1,v_2,....v_n; v_{n+1},v_{n+2},....v_{2n} will form a spanning path in G. To define f :V $\rightarrow \{0, 1, 2, ...q\}$ the following four cases are to be considered.

Case (i) $n \equiv 0(mod\ 4)$

For $i \leq \frac{n}{2}-1$

$f(v_i) = \frac{n+i+1}{2}$; when i is odd

$= \frac{3}{2}n - (\frac{i}{2}-1)$; when i is even

For $\frac{n}{2}$ ≤i≤ $n-1$

$f(v_i) = \frac{3}{2}n - \frac{i}{2}$; when i is even

$= \frac{n+i+1}{2}$; when i is odd

For $i = n$; $f(v_i) = 0$

For $n + 1 \leq i \leq \frac{3}{2}n$

$f(v_i)$=$(2n + 1)$-($\frac{i-n-1}{2}$); when i is odd

$= \frac{i-n}{2}$; when i is even

For $\frac{3n+2}{2} \leq i \leq 2n$

$f(v_i)$=$(2n + 1) - (\frac{i-n+1}{2})$; when i is odd

$= \frac{i-n}{2}$; when i is even

Case (ii) $n \equiv 1(mod\ 4)$

$f(v_1)$=0

For 2≤i≤ $\frac{n-1}{2}$

$f(v_i) = n + (\frac{i+1}{2})$; when i is odd

$= (n + 1) - (\frac{i}{2})$; when i is even

For $\frac{n+1}{2} \leq i \leq n - 1$

$f(v_i) = (n + 1) - (\frac{i+1}{2})$; when i is odd

$= n + (\frac{i+2}{2})$; when i is even

For $n \leq i \leq 2n$

$f(v_i) = (2n + 1)$-($\frac{i-n}{2}$) ; when i is odd

=$\frac{i+1-n}{2}$; when i is even

Case (iii) $n \equiv 2(mod\ 4)$

For 1≤ $i \leq n - 2$

$f(v_i)$=$\frac{i}{2}$; when i is even

$= (2n + 1) - (\frac{i-1}{2})$; when i is odd

$f(v_{n-1}) = f(v_{n-3}) - f(v_{n-2}) - 1$

$f(v_n) = 0$

For $n + 1 \leq i \leq \frac{3n}{2} - 1$

$f(v_i)$=$\frac{i-1}{2}$; when i is odd

$= (2n + 1)$-($\frac{i-4}{2}$); when i is even

$f(v_i)$=$\frac{i+1}{2}$; for i=$\frac{3n}{2}$

For $\frac{3n}{2} + 1 \leq i \leq \frac{3n}{2} + 2$

$f(v_i)$=$\frac{i+2}{2}$; when i is even

$= (2n + 1)$-($\frac{i-5}{2}$); when i is odd

For $\frac{3n}{2}+3 \leq i \leq 2n$

$f(v_i)$=$(2n+1)$-$(\frac{i-5}{2})$; when i is odd

=$\frac{i+4}{2}$; when i is even

Case (iv) $n \equiv 3(mod\ 4)$

$1 \leq i \leq \frac{n-1}{2}$

$f(v_i)$=$\frac{3(n+1)}{2}-(\frac{i+1}{2})$; when i is odd

$=\frac{n+i+1}{2}$; when i is even

For$\frac{n+1}{2} \leq i \leq n-2$

$f(v_i)$=$\frac{3(n+1)}{2}-(\frac{i+2}{2})$; when i is even

$=\frac{n+i+2}{2}$; when i is odd

$f(v_{n-1})=0$

For $n \leq i \leq 2n$

$f(v_i)$=$(2n+1)$-$(\frac{i-n}{2})$; when i is odd

=$\frac{i+1-n}{2}$; when i is even

In the above four cases it is possible to assign labels in such a way that it provides graceful labeling for the joint sum of two copies of cycle C_n. ∎

Illustration 2.6. *The joint sum of two copies of* C_{13} *is as shown in Figure 6.*

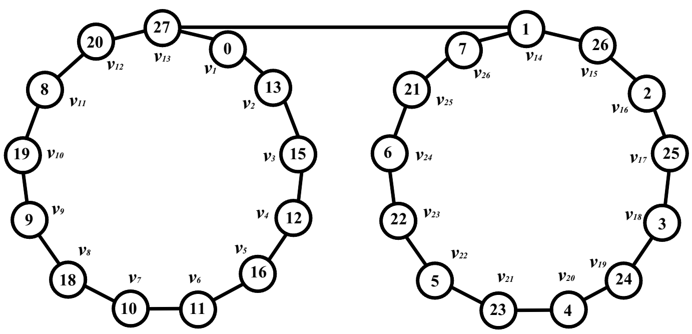

Figure 6: Graceful labeling of the joint sum of two copies of C_{13}

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