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Hamiltonian partition coloring

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Abstract

The hamiltonian coloring of a connected graph G introduced by Chartrand et al [1] is different from hamiltonian partition coloring. In this paper, we characterize graphs which has a hamiltonian partition. Also, we give example of graphs having prescribed chromatic numbers and hamiltonian partition numbers. We derive results connecting the hamiltonian chromatic number of $G_1 \cup G_2$ and $G_1 + G_2$.

Key words: Hamiltonian partition, Hamiltonian partition number.

AMS Subject Classification(2010): 05C15.

1 Introduction

Prof. E. Sampathkumar and Dr. V. N. Bhave [6] have defined partition graph of a graph as follows:

Given a graph $G = (V, E)$ and a partition $P = \{V_1, V_2, \cdots, V_s\}$ of $V(G)$. The partition graph $P(G)$ of P has P as it point set and V_i and V_j are adjacent if and only if there exists $v_i \in V_i$ and $v_j \in V_j$ such that v_i and v_j are adjacent in G. $P(G)$ is a homomorphic image of G if every set in P is independent in G. $P(G)$ is a contraction of G if every set in P induces a connected subgraph in G. In the first case P is called a homomorphism and in the second case P is called a contraction. A partition P of $V(G)$ is said to be n-complete if $P(G) = K_n$. It is easily seen that the chromatic number $\chi(G)$ is the minimum n for which G has an n-complete homomorphic partition in which every element of P is independent and the achromatic number $\psi(G)$ is the maximum number n for which G has an n-complete homomorphic partition in which every element of P is independent.

2 Main Results

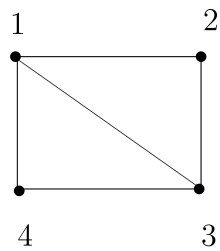
Definition 2.1. *A partition P of $V(G)$ is called proper color partition if every element of P is an independent set of G.*

Definition 2.2. *A proper color partition P of $V(G)$ is called a hamiltonian partition if $P(G)$ is hamiltonian.*

Remark 2.3. *Every Chromatic as well as achromatic partition of G of cardinality ≥ 3 is a hamiltonian partition. The converse is not true.*

For example, consider K_4 - $\{e\}$.

Example 2.4.



The partition $\{\{1\},\{2\},\{3\},\{4\}\}$ of $K_4 - \{e\}$is a hamiltonian partition but is neither a chromatic partition nor an achromatic partition.

Definition 2.5. *The maximum cardinality of a hamiltonian partition coloring of G is called the hamiltonian partition achromatic number of G and is denoted by $\chi_h(G)$.*

Remark 2.6. $\chi(G) \leq \psi(G) \leq \chi_h(G)$.

Observation 2.7. *Any partition graph of $K_{1,n}$ is a star.*

Observation 2.8. *Let G be a connected graph which does not contain any subgraph isomorphic to P_4 or C_3. Then G is a star.*

Observation 2.9. *For a given positive integer k there exist graphs G for which $\chi_h(G) - \chi(G) = k$.*

Proof. Let G be a path of order $k+3$. Then $\chi(G)$ = 2 and $\chi_h(G) = k+2$. ∎

Observation 2.10. *G is a hamiltonian graph if and only if $\chi_h(G) = n$.*

Theorem 2.11. *Let G be a graph without isolates. Then $\chi_h(G) = n-1$ if and only if G is not hamiltonian but has a hamiltonian path.*

Proof. If G has a hamiltonian path say $u_1u_2\cdots u_n$, then
$V_1 = \{u_1, u_n\}$, $V_2 = \{u_2\}$,$\cdots$,$V_{n-1} = \{u_{n-1}\}$ is a cyclic partition of G. Therefore, $\chi_h(G) \geq n-1$. Since G is not hamiltonian $\chi_h(G) \leq n-1$. Therefore, $\chi_h(G) = n-1$.
Conversely, suppose $\chi_h(G) = n-1$. Then there exists a partition of $V(G)$ into $V_1, \cdots, V_{n-1}$ such that the partition graph of $\{V_1, V_2, \cdots, V_{n-1}\}$ is hamiltonian. Since $|V(G)| = n$, exactly one V_i has two elements and all others are singletons. Let V(G) = $\{u_1, u_2, \cdots, u_n\}$. Let without loss of generality $V_1 = \{u_1, u_n\}$,$V_2 = \{u_2\}$, $\cdots$, $V_{n-1} = \{u_{n-1}\}$. Since the partition graph is hamiltonian without loss of generality it can be assumed that u_i is adjacent to u_{i-1},u_{i+1} ; $2 \leq i \leq n-3$, u_1 is adjacent to u_2 or u_n is adjacent to u_2; u_{n-1} is adjacent to u_1 or u_n.
Case (i) u_{n-1} is adjacent to u_1 and u_1 is adjacent with u_2. Since u_n is not an isolate, u_n is adjacent to some u_i, $1 \leq i \leq n-1$. Then $u_nu_iu_{i+1}\cdots u_{n-1}u_1u_2\cdots u_{i-1}$ is a hamiltonian path .
Case (ii) u_{n-1} is adjacent to u_n and u_1 is adjacent with u_2. Then we have the path $u_1u_2\cdots u_{n-1}u_n$.
Since $\chi_h(G) = n-1$, G is not hamiltonian.
Case (iii) u_{n-1} is adjacent with u_1 and u_n is adjacent with u_2. Then $u_nu_2\cdots u_{n-1}u_1$ is a hamiltonian path.
Since $\chi_h(G) = n-1$, G is not hamiltonian.
Case (iv) u_{n-1} is adjacent with u_n and u_n is adjacent with u_2. u_n being a non-isolate is adjacent with some $u_j,\ 1 \leq j \leq n-1$. Then $u_1u_j\cdots u_{n-1}u_nu_2u_3\cdots u_{j-1}$ is a hamiltonian path.
Since $\chi_h(G) = n-1$, G is not hamiltonian. ∎

Observation 2.12. *Let G be a graph with t isolates say $u_1, u_2, \cdots, u_t$. Then $\chi_h(G)$ = $\chi_h(G - \{u_1, \cdots, u_t\})$.*

Theorem 2.13. *Let G be a simple connected graph. Then G has a hamiltonian partition if and only if G is not a star.*

Proof. Let G be a simple connected graph. To prove the theorem it is enough to show that G has a hamiltonian partition if and only if G has a subgraph isomorphic to P_4 or C_3. For:
$\mathcal{A}$: Suppose G has a subgraph isomorphic to P_4. Let u_1, u_2, u_3, u_4 be the vertices in

G where $u_1u_2, u_2u_3, u_3u_4 \in E(G)$.

Case (1) u_1 is adjacent to u_4.

Then take $V_1 = \{u_1\}$, $V_2 = \{u_2\}$, $V_3 = \{u_3\}$, $V_4 = \{u_4\}$.

Let $H = G - \{u_1, u_2, u_3, u_4\}$. Let $P = \left\{U_1, \cdots, U_{\chi(H)}\right\}$ be a chromatic partition of H.

Subcase (i) $\chi(H) \geq 3$.

Subsubcase (i) Suppose the subgraph induced by u_1, u_2, u_3, u_4 is a component of G. If $\chi(H) = 3$, then the three classes of P can be merged with V_1, V_2, V_3 and V_4. If $\chi(H) \geq 4$ then V_1, V_2, V_3 and V_4 can be merged with elements of P.

Subsubcase (ii) Suppose the subgraph induced by u_1, u_2, u_3, u_4 is not a component of G. Let without loss of generality u_4 is adjacent to some vertex say w in U_1. If u_1 is adjacent to some class U_i, $i \neq 1$, then $\{u_4\}U_1, \cdots U_i \cdots U_\chi \hat{U}_i$

$\{u_1\}\{u_2\}\{u_3\}\{u_4\}$ is a cycle.

If u_1 is adjacent to U_1 and u_1 is not adjacent to any $U_i(2 \leq i \leq \chi)$, then add u_1 with U_2 . Then $\{u_4\}U_1\hat{U}_2 \cdots U_\chi U_2\{u_2\}\{u_3\}\{u_4\}$ is a cycle.

Subcase (ii) $\chi(H) \leq 2$. The arguments given in the subcase (i) can be repeated.

Case (2) Suppose u_1 is adjacent to u_3

Subcase (i) Suppose the subgraph induced by u_1, u_2, u_3, u_4 is a component of G. Let $V_1 = \{u_1u_4\}, V_2 = \{u_2\}, V_3 = \{u_3\}$.

If $\chi(H) \geq 3$, then V_1, V_2, V_3 can be merged with elements of P. If $\chi(H) = 2$, then $U_1 \cup V_1$, $U_2 \cup V_2$, V_3 is a cycle. If $\chi(H) = 1$, then $U_1 \cup V_1$, V_2, V_3 is a cycle.

Subcase (ii) The subgraph induced by u_1, u_2, u_3, u_4 is not a component of G. Let $\chi(H) \geq 3$.

A: u_4 is adjacent to a vertex say w in U_1.

A_1 : If u_1 is adjacent to some U_i, $i \neq 1$, then $\{u_4\}U_1 \cdots, \hat{U}_i, \cdots, U_\chi U_i$ $\{u_1\}\{u_2\}\{u_3\}\{u_4\}$ is a cycle.

A_2: If u_1 is adjacent to U_1 and is not adjacent to any U_i, then add u_1 to U_χ giving U'_χ. Now $\{u_4\}U_1 \cdots U'_\chi\{u_2\}\{u_3\}\{u_4\}$ is a cycle.

B: u_4 is not adjacent to any U_i, $1 \leq i \leq \chi$. Then u_1 or u_2 or u_3 is adjacent to some U_i.

B_1: Let u_1 is adjacent to say U_1, add u_4 with U_χ giving U'_χ. Then $\{u_1\}U_1 \cdots U'_\chi\{u_3\}\{u_2\}\{u_1\}$ is a cycle.

B_2: If u_2 is adjacent to say U_1, then $\{u_2\}U_1 \cdots U^1_\chi\{u_3\}\{u_1\}\{u_2\}$ is a cycle.

B_3: u_3 is adjacent to say U_1.

B_{3_1} : u_1 is adjacent to some U_i, $i \neq 1$. Add u_4 to U_χ giving U'_χ. Then $\{u_3\}U_1 \cdots \hat{U}_i \cdots U^1_\chi U_i\{u_1\}\{u_2\}\{u_3\}$ is a cycle.

B_{3_2} : u_1 is adjacent to U_1 and u_1 is not adjacent to any U_i, $2 \leq i \leq \chi$. Add u_1, u_4 with U_χ giving U''_χ. Then $\{u_3\}U_1 \cdots U''_\chi\{u_2\}\{u_3\}$ is a cycle. Let $\chi(H) = 2$.

A' : u_4 is adjacent to U_1.

A'_1 : u_1 is adjacent to U_2 then $\{u_4\}U_1U_2\{u_1\}\{u_2\}\{u_3\}\{u_4\}$ is a cycle.

A'_2 : u_1 is adjacent to U_1 but not adjacent to U_2. Add u_1 with U_2 giving U'_2. Then $\{u_4\}U_1U'_2\{u_2\}\{u_3\}\{u_4\}$ is a cycle.

B' : u_4 is adjacent not adjacent to U_1, U_2.

B'_1 : u_1 is adjacent to U_1. Add u_4 with U_2 giving U'_2.

Then $\{u_1\}U_1U'_2\{u_3\}\{u_2\}\{u_1\}$ is a cycle.

B'_2 : u_2 is adjacent to U_1. Add u_4 with U_2 giving U'_2.

Then $\{u_2\}U_1U'_2\{u_3\}\{u_1\}\{u_2\}$ is a cycle.

B'_3 : u_3 is adjacent to U_1. Add u_4 with U_2 giving U^1_2.

B'_{3_1} : u_1 is adjacent to U_2. Then $\{u_3\}U_1U'_2\{u_1\}\{u_2\}\{u_3\}$ is a cycle.

B'_{3_2} : u_1 is not adjacent to U_2. Add u_1, u_4 with U_2 giving U''_2.
Then $\{u_3\}U_1U''_2\{u_2\}\{u_3\}$ is a cycle.
Let $\chi(H) = 1$.

A'' : u_4 is adjacent to U_1.

A''_1 : u_1 is adjacent to U_1. Then $\{u_4\}U_1\{u_1\}\{u_2\}\{u_3\}\{u_4\}$ is a cycle.

A''_2 : u_1 is not adjacent to U_1. Add u_1 with U_1 giving U'_1.
Then $\{u_4\}U'_1\{u_2\}\{u_3\}\{u_4\}$ is a cycle.

B'' : u_4 is not adjacent to U_1. Add u_4 with U_1 giving U'_1.

B''_1 : u_1 is adjacent to U_1. Then $U'_1\{u_1\}\{u_2\}\{u_3\}U'_1$ is a cycle.

B''_2 : u_2 is adjacent to U_1. Then $U'_1\{u_2\}\{u_1\}\{u_3\}U'_1$ is a cycle.

B''_3 : u_3 is adjacent to U_1.

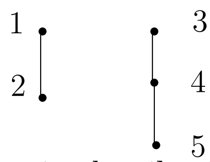
B''_{3_1} : u_1 is adjacent to U_1. Then $U'_1\{u_1\}\{u_2\}\{u_3\}U'_1$ is a cycle.

B''_{3_2} : u_1 is not adjacent to U_1. Add u_1, u_4 with U_1 giving U''_1.
Then $\{u_3\}U''_1\{u_2\}\{u_3\}$ is a cycle.
$\mathcal{B}$: Suppose G contains a cycle C_3. This case is similar to Case(1) of $\mathcal{A}$ and the result follows.

Conversely, suppose G has a hamiltonian partition and G is connected. Suppose G has no subgraph isomorphic to P_4 or C_3. Then G is $K_{1,n}$ for some $n \geq 1$. By observation, any partition of $K_{1,n}$ is a star, a contradiction. Therefore, G has a subgraph isomorphic to P_4 or C_3. ∎

Remark 2.14. *If G is disconnected, then the converse of the theorem need not be true. That is, a disconnected graph may have a hamiltonian partition though it contains no subgraph isomorphic to P_4 or C_3.*

Example 2.15.



Let π = $\{\{1,3\},\{4\},\{2,5\}\}$. Then π is a hamiltonian partition.

Theorem 2.16. *Let G be a disconnected graph. Let $|E(G)| \geq 3$ and $G \neq K_{1,n} \cup tK_1, t \geq 1, n \geq 3$, then G has a hamiltonian partition.*

Proof. If G has a subgraph isomorphic to P_4 or C_3 then G has a hamiltonian partition. Suppose G does not contain any subgraph isomorphic to P_4 or C_3. Let $|E(G)| \geq 3$ and $G \neq K_{1,n} \cup tK_1$, $t \geq 1$.
Let $G_1, G_2, \cdots, G_k$ be the components of G. Then by hypothesis each G_i is a star. Since $|E(G)| \geq 3$, and $G \neq K_{1,n} \cup tK_1$, G contains either $3K_2$ or $K_2 \cup K_{1,2}$ or $K_{1,n} \cup K_{1,t}$ $(n \geq 3, t \geq 1)$.
Case (i) G contains $3K_2$. Let $V = \{1,2,3,4,5,6\}$ be the vertex set of $3K_2$ with 1 adjacent with 2,3 adjacent with 4 and 5 adjacent with 6. Let $V_1 = \{1,6\}, V_2 = \{2,3\}, V_3 = \{4,5\}$. Other components can be suitably merged with V_1 and V_2. Let $V_1^{'}$, $V_2^{'}$ and $V_3^{'}$ be the resulting partition of $V(G)$. Then $V_1^{'}$ is adjacent with $V_2^{'}$ (1 is adjacent with 2),$V_2^{'}$ is adjacent with $V_3^{'}$ (3 is adjacent with 4)and $V_3^{'}$ is adjacent with $V_1^{'}$ (5 is adjacent with 6). Hence G contains a hamiltonian partition.
Case (ii) G contains $K_2 \cup K_{1,2}$. Let $V = \{1,2,3,4,5\}$ be the vertex set of $K_2 \cup K_{1,2}$ with 1 adjacent to 2, 3 adjacent with 4 and 5. Let $V_1 = \{1,4\}, V_2 = \{3\}, V_3 = \{2,5\}$. Other components can be suitably merged with V_1 and V_2. Let $V_1^{'}$, $V_2^{'}$ and $V_3^{'}$ be the resulting partition of $V(G)$. Then $V_1^{'}$ is adjacent with $V_2^{'}$ (4 is adjacent with 3),$V_2^{'}$ is adjacent with $V_3^{'}$ (3 is adjacent with 5)and $V_3^{'}$ is adjacent with $V_1^{'}$ (2 is adjacent with 1). Hence G contains a hamiltonian partition.
Case (iii) G contains $K_{1,n} \cup K_{1,t}$, $(n \geq 3, t \geq 1)$.
Let $V = \{v, u_1, u_2, u_3, \cdots, u_n\}$ be the vertex set of $K_{1,n}$ with v adjacent with

$u_1, u_2, u_3, \cdots, u_n$ and $V^{'} = \{w, x_1, \cdots, x_t\}$ be the vertex set of $K_{1,t}$ with w adjacent with $x_1, \cdots, x_t$. Let $V_1 = \{v\}$, $V_2 = \{u_1, w\}$, $V_3 = \{u_2, \cdots, u_n, x_1, \cdots, x_t\}$. Other components can be suitably merged with V_1 and V_2. Let $V_1^{'}, V_2^{'}$ and $V_3^{'}$ be the resulting partition of $V(G)$. Then $V_1^{'}$ is adjacent with $V_2^{'}$ (v is adjacent with u_1),$V_2^{'}$ is adjacent with $V_3^{'}$ (w is adjacent with x_1)and $V_3^{'}$ is adjacent with $V_1^{'}$ (u_2 is adjacent with v).
Hence G contains a hamiltonian partition. ∎

The preceding theorems and observations lead to the following theorem.

Theorem 2.17. *Let G be a simple graph. G has a hamiltonian partition if and only if $G \neq K_{1,n} \cup tK_1$ ($n \geq 0, t \geq 0$), $2K_2 \cup tK_1$ ($t \geq 0$).*

Theorem 2.18. *For every two positive integers a and b with $3 \leq a < b - 1$ there exists a graph G with $\chi(G) = a$ and $\chi_h(G) = b$.*

Proof. Let $r = b-a+1 \geq 3$. Let $G = K_a \cup rK_2$. Then $\chi(G) = a$. Consider the partition $\pi = \left\{\left\{u_1, v_r^{'}\right\}, \{u_2\}, \cdots, \{u_a, v_1\}, \{v_2, v_1^{'}\}, \{v_3, v_2^{'}\}, \cdots, \{v_r, v_{r-1}^{'}\}\right\}$ where $V(K_a) = \{u_1, \cdots, u_a\}$ and $V(rK_2) = \{v_1, v_1^{'}, v_2, v_2^{'}, \cdots, v_r, v_r^{'}\}$ where v_i is adjacent with $v_i^{'}$, $1 \leq i \leq r$. Clearly, π is a hamiltonian partition. Therefore, $\chi_h(G) \geq a + r - 1$, $\chi_h(K_a) = a$ and $\chi_h(rK_2) = r$. Each partite set of rK_2 is a doubleton set. Hence at most $a + r - 1$ partite classes may exist in $K_a \cup rK_2$ forming a hamiltonian cycle. Therefore, $\chi_h(K_a \cup rK_2) \leq a + r - 1$. Thus $\chi_h(K_a \cup rK_2) = a + r - 1 = b$. ∎

Remark 2.19. *If $3 \leq a < b - 1$, there exists a graph G with $\psi(G) = a$ and $\chi_h(G) = b$ (The graph in the above theorem serves the purpose).*

Definition 2.20. *Let G be a graph for which the partition graph has a spanning path. The maximum order of a partition graph of G which has a spanning path is called the hamiltonian path partition of G and is denoted by $\chi_{hp}(G)$.*

Theorem 2.21. *Let G be a graph having hamiltonian partition. Then $\chi_h(G) \leq \chi_{hp}(G)$
$\leq \chi_h(G) + 1$.*

Proof. Let $\chi_h(G)$ = k and $\chi_{hp}(G)$ = l. Therefore, $l \geq k$. suppose $l \geq k+2$. Let $\{V_1, \cdots, V_l\}$ be a maximum hamiltonian path partition of G. Suppose there exists an edge between V_1 and V_l. Then there is a hamiltonian partition of cardinality l. Therefore, k = $\chi_h(G) \geq l \geq k+2$, a contradiction. Therefore, there exists no edge between V_1 and V_l. Let $V_1^{'}$ = $V_1 \cup V_l$. Then $\{V_1^{'}, V_2^{'}, \cdots, V_{l-1}^{'}\}$ is a hamiltonian partition. Therefore, $k = \chi_h(G) \geq l-1 \geq k+1$, a contradiction. Therefore, $l \leq k+1$ and hence $\chi_h(G) \leq \chi_{hp}(G) \leq \chi_h(G)$ + 1. ∎

Remark 2.22. *(i) If $G = P_4$,then $\chi_h(G)$ = 3, $\chi_{hp}(G)$ = 4.*
(ii) If $G = K_n$, $\chi_h(G)$ = n = $\chi_{hp}(G)$.

Result 2.23. $\chi_h(G_1 \cup G_2) \leq \chi_h(G_1)$+$\chi_h(G_2) \leq \chi_h(G_1 \cup G_2)$ *+ 2.*

Proof. Let $\chi_h(G_1 \cup G_2)$ = t. Let $\{V_1, V_2, \cdots, V_t\}$ be the maximum hamiltonian partition of $G_1 \cup G_2$. There are t edges in a hamiltonian cycle of the partition graph formed by $V_1, V_2, \cdots, V_t$. Of the t edges, let x be the number of edges in G_1 and y be the number of edges in G_2. Therefore, $x+y=t$. Therefore, $V(G_1)$ can be partitioned into $x+1$ classes such that the partition graph of this partition has a hamiltonian path. Likewise, $V(G_2)$ can be partitioned into y + 1 classes such that the partition graph of this partition has a hamiltonian path. If there exists a hamiltonian path partition in G_1 of order $s \geq x+2$, then $\chi_h(G_1 \cup G_2) \geq x+1+y = t+1$, a contradiction. A similar argument shows that there cannot be a hamiltonian path partition in G_2 of order $\geq y+2$. Therefore, $\chi_{hp}(G_1) = x+1$, $\chi_{hp}(G_2) = y+1$ and hence $\chi_h(G_1) = x$ (or) $x+1$, $\chi_h(G_2) = y$ (or) $y+1$. Therefore, $\chi_h(G_1)$ + $\chi_h(G_2) = x+y$ or $x+y+1$ or $x+y+2$. Thus $\chi_h(G_1) + \chi_h(G_2) = t$ or $t+1$ or $t+2$. ∎

Theorem 2.24. *Let G_1 and G_2 be two vertex disjoint graphs with hamiltonian partitions. Suppose there exists a hamiltonian partition of maximum cardinality in G_2 say $\left\{W_1, \cdots, W_{\chi_h(G_2)}\right\}$ satisfying the following:*

(i) There exists two edges between W_i and W_{i+1}, $1 \leq i \leq \chi_h(G_2) - 1$.

(ii) If x_1y_1 and x_2y_2 are the edges between W_i and W_{i+1}, then

(a) If $y_1 \neq y_2$, then there exists an edge uv from W_{i-1} to W_i with $u \in W_{i-1}$, $v \neq x_1, x_2$.

(b) If $y_1 = y_2$ *then there exists an edge uv between* W_{i+1} *and* W_{i+2} *such that* $u \in W_{i+1}$ *and* $u \neq y_1$.

Then $\chi_h(G_1 \cup G_2) = \chi_h(G_1) + \chi_h(G_2)$.

Proof. Suppose the conditions in the theorem are satisfied. Let $\{V_1, V_2, \cdots, V_{\chi_h(G_1)}\}$ be a hamiltonian partition of maximum cardinality in G_1. Let x_1y_1 and x_2y_2 be two edges between W_i and W_{i+1} (x_1 may be equal to x_2 or y_1 may be equal to y_2).

Case (i) Suppose $y_1 \neq y_2$. Add y_1 with $V_{\chi_h(G_1)}$. By (ii) (a) there exists an edge uv from W_{i-1} to W_i with $v \neq x_1, x_2$. Add v with V_1. Then,
$\pi = \left\{V_1, V_2, \cdots, V_{\chi_h(G_1)}, W_i, W_{i+1}, \cdots, W_{i-1}\right\}$ is a hamiltonian partition of $G_1 \cup G_2$.

Case (ii) Suppose $y_1 = y_2$. Add x_1 with $V_{\chi_h(G_1)}$. Consider the partition $\pi_1 = \left\{V_1, V_2, \cdots, V_{\chi_h(G_1)}, W_{i+1}, W_i, W_{i-1}, \cdots, W_{i+2}\right\}$. By (ii) (b) there exists an edge uv between W_{i+1} and W_{i+2} such that $u \in W_{i+1}$ and $u \neq y_1$. Add u with V_1. Then π_1 is a hamiltonian partition in $G_1 \cup G_2$. Thus in either case, $\chi_h(G_1 \cup G_2) \geq \chi_h(G_1) + \chi_h(G_2)$. But $\chi_h(G_1 \cup G_2) \leq \chi_h(G_1) + \chi_h(G_2)$.
Therefore, $\chi_h(G_1 \cup G_2) = \chi_h(G_1) + \chi_h(G_2)$. ∎

Observation 2.25. $\chi_h(G_1 \cup G_2) \geq \chi_h(G_1) + \chi_h(G_2) - 2$.

For:
Let $\pi_1 = \{S_1, \cdots, S_{\chi_h(G_1)}\}$ be a χ_h-partition of G_1 and $\pi_2 = \{T_1, \cdots, T_{\chi_h(G_2)}\}$ be a χ_h-partition of G_2. Then $\pi_3 = \{S_1 \cup T_{\chi_h(G_2)}, S_2, \cdots, S_{\chi_h(G_1)} \cup T_1, \cdots, T_{\chi_h(G_2)-1}\}$ is a hamiltonian partition of $G_1 \cup G_2$. Therefore, $\chi_h(G_1 \cup G_2) \geq \chi_h(G_1) + \chi_h(G_2) - 2$.

Theorem 2.26. *Let* G_1 *and* G_2 *be two vertex disjoint simple graphs with hamiltonian partitions. Suppose for any hamiltonian partition* $\pi_1 = \{V_1, V_2, \cdots, V_{\chi_h(G_1)}\}$ *of* G_1 *and* $\pi_2 = \{W_1, W_2, \cdots, W_{\chi_h(G_2)}\}$ *of* G_2 *there exists exactly one edge between* V_i *and* V_{i+1}, $1 \leq i \leq \chi_h(G_1)$ *and* W_j *and* W_{j+1}, $1 \leq j \leq \chi_h(G_2)$.

(a) Suppose the edge joining V_1 *and* V_2 *and the edge joining* $V_{\chi_h(G_1)}$ *and* V_1 *are not adjacent or a similar condition holds in* π_2*. Then* $\chi_h(G_1 \cup G_2) = \chi_h(G_1) + \chi_h(G_2)$ *- 1.*

(b) *Suppose the edge joining V_1 and V_2 and the edge joining $V_{\chi_h(G_1)}$ and V_1 are adjacent and the similar conditions hold in π_2 also. Then $\chi_h(G_1 \cup G_2)$ = $\chi_h(G_1)$ + $\chi_h(G_2)$ - 2.*

Proof. (a) Let v_1v_2 be the edge between V_1 and V_2 ($v_1 \in V_1, v_2 \in V_2$). Let v_tv_s be the edge between $V_{\chi_h(G_1)}$ and V_1 where $v_t \in V_{\chi_h(G_1)}$ and $v_s \neq v_1$. Then join v_1 with $W_{\chi_h(G_2)}$ and v_s with W_1 . Add v_1 with $W_{\chi_h(G_2)}$ resulting in $W^{'}_{\chi_h(G_2)}$.
Consider $\pi_3 = \left\{W_1, W_2, \cdots, W^{'}_{\chi_h(G_2)}, V_2, \cdots, V_{\chi_h(G_1)}\right\}$. π_3 is a hamiltonian partition of $G_1 \cup G_2$. Therefore, $\chi_h(G_1 \cup G_2) \geq \chi_h(G_1) + \chi_h(G_2)$ - 1. By the condition in (a) of the theorem, $\chi_h(G_1 \cup G_2) < \chi_h(G_1)$ + $\chi_h(G_2)$. Therefore, $\chi_h(G_1 \cup G_2)$ = $\chi_h(G_1)$ + $\chi_h(G_2)$ - 1.

(b) Let v_1v_2 be the edge between V_1 and V_2 ($v_1 \in V_1, v_2 \in V_2$). By the condition in (b), the edge from $V_{\chi_h(G_1)}$ to V_1 is incident with v_1. Add every vertex of $V_{\chi_h(G_1)}$ with W_1 resulting in $W^{'}_1$. Then $\pi_4 = \{W^{'}_1, \cdots, W_{\chi_h(G_2)}, V_2, V_3, \cdots,$ $V_{\chi_h(G_1)-1}\}$ is a hamiltonian partition of $G_1 \cup G_2$. Therefore, $\chi_h(G_1 \cup G_2) \geq$ $\chi_h(G_1) + \chi_h(G_2)$ - 2. By the condition in (b) of the theorem , $\chi_h(G_1 \cup G_2) < \chi_h(G_1)$ + $\chi_h(G_2)$ - 1.
Therefore, $\chi_h(G_1 \cup G_2)$ = $\chi_h(G_1)$ + $\chi_h(G_2)$ - 2. ∎

Theorem 2.27. *Let G be a graph with hamiltonian partition. Let $\pi = \{S_1, \cdots, S_k\}$ be a χ_h-partition of G. Let the edge from S_1 to S_2 used in the hamiltonian cycle be u_1u_2. Attach pendent vertices u_{k+1}, u_{k+2} at u_1, u_2 respectively. Let H be the resulting graph. Then $\chi_h(H) = \chi_h(G) + 1$.*

Proof. Let $\pi_1 = \{S_1, S^{'}_1, S_2, \cdots, S_k\}$ where $S^{'}_1 = \{u_{k+1}, u_{k+2}\}$. Then π_1 is hamiltonian partition of H. Therefore, $\chi_h(H) \geq \chi_h(G)$+ 1. Suppose $\chi_h(H) \geq \chi_h(G)$+ 2. Let $\pi_2 = \{T_1, \cdots, T_l\}$ be a χ_h-partition of H. Then $l \geq \chi_h(G)$+ 2. Suppose u_{k+1} and u_{k+2} belong to different sets of π_2. Since u_{k+1} and u_{k+2} have degree one, none of them can be used in the hamiltonian cycle. Hence the vertices used to form the hamiltonian cycle will be from G. Therefore, $\chi_h(G) \geq l \geq \chi_h(G)$ + 2, a contradiction. If u_{k+1} and u_{k+2} are in the same set of π_2 and if both are used in the hamiltonian cycle, then $\chi_h(H) = \chi_h(G)$ + 1. ∎

Theorem 2.28. *Let $|V(G_1)| = n, \ |V(G_2)| = m$ and let $n \geq m$. Then $\chi_h(G_1 + G_2) = 2\,|V(G_2)| + t - 1$ where t is the maximum of hamiltonian path partition of* *subgraphs of G_1 of order $n-m+1$.*

Proof. Let $V(G_1)=\{u_1,\cdots,u_n\}$ and $V(G_2)=\{v_1,\cdots,v_m\}$. Let $\pi=\{S_1,S_2,\cdots,S_{2m}\}$ where $S_1=\{u_1\},S_2=\{v_1\},\cdots,S_{2m}=\{v_m\}$. Then π is a hamiltonian partition of G_1+G_2. Let t be the maximum of hamiltonian path partition of subgraphs of G_1 of ordern $n-m+1$. Let H be such a subgraph and let $V(H)=\{x_1,x_2,\cdots,x_{n-m+1}\}$. Let $\pi_1=\{T_1,T_2,\cdots,T_t\}$ be a χ_{hp}-partition of H. Let $V(G_1)-V(H)=\{y_1,y_2,\cdots,y_{m-1}\}$.
Let $\pi_2=\{T_1,T_2,\cdots,T_t,\{v_1\},\{y_1\},\cdots,\{v_{m-1}\},\cdots,\{v_m\}\}$. Then π_2 is a hamiltonian partition of G_1+G_2. $|\pi_2|=t+2m-1=2\,|V(G_2)|+t-1$. Therefore, $\chi_h(G_1+G_2)\geq 2\,|V(G_2)|+t-1$. Let $\pi_3=\{W_1,W_2,\cdots,W_s\}$ be a χ_h - partition of G_1+G_2. Then $|\pi_3|\geq 2m$. Further the sets in π_3 which are not singleton must form a hamiltonian path. Suppose there are $2m$ singleton sets in π_3 and the remaining sets are without loss of generality $W_1,W_2,\cdots,W_{s-2m}$. Therefore, $\pi_3=\{W_1,W_2,\cdots,W_{s-2m},\{v_1\},\{y_1\},\cdots,\{v_m\},\{y_m\}\}$ where $\{y_1,\cdots,y_m\}=V(G_1)-(W_1\cup W_2\cdots\cup W_{s-2m})$. Then $\{y_m\}W_1W_2\cdots,W_{s-2m}$ is a hamiltonian path in a subgraph of G_1 of order $s-2m+1$. $s-2m+1=\chi_h(G_1+G_2)-2m+1$. But $s-2m+1\leq t$. Therefore, $\chi_h(G_1+G_2)-2m+1\leq t$. Hence $\chi_h(G_1+G_2)\leq 2m+t-1$. Thus $\chi_h(G_1+G_2)=2m+t-1=2\,|V(G_2)|+t-1$. ∎

Observation 2.29. *Let $\pi_1=\{S_1,S_2,\cdots,S_k\}$ be a hamiltonian partition of G and let there exist two edges $e_1=u_1v_1$ and $e_2=u_2v_2$ between S_{i-1} and S_i (for some $i,2\leq i\leq k$) such that $v_1\neq v_2$ (u_1 may be equal to u_2) and there exists an edge v_1y or v_2y from S_i to S_{i+1}. Let there exists an edge w_1w_2 from S_{i-2} toS_{i-1} with $w_2\neq u_1,\ u_2$. Then there exists a hamiltonian partition π_2 of G such that $|\pi_2|>|\pi_1|$.*

Proof. Let $T_1=\{u_1,u_2\}$. Let $\pi_2=\{T_1,S_i,S_{i+1},\cdots,(S_{i-1}-\{u_1,u_2\})\cup\{v_\alpha\}\}$. where $\alpha=\begin{cases}1 & \text{if } T_2=\{v_2\}\\ 2 & \text{if } T_2=\{v_1\}\end{cases}$ Then π_2 is a hamiltonian partition of G (Since there exists an edge from S_{i-2} *to* $(S_{i-1}-\{u_1,u_2\})\cup\{v_\alpha\})$ and $|\pi_2|=|\pi_1|+1$. ∎

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