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Some new odd harmonious graphs

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Abstract

A graph which admits odd harmonious labeling is called an odd harmonious graph. In this paper we prove that the shadow graphs of path P_n and star $K_{1,n}$ are odd harmonious. Further we prove that the split graphs of path P_n and star $K_{1,n}$ admit odd harmonious labeling.

Keywords: Harmonious labeling, Odd harmonious labeling, Shadow graph, Split Graph.

AMS Subject Classification (2010): 05C78.

1 Introduction

Throughout this paper we consider simple, finite, connected and undirected graph $G = (V(G), E(G))$ with p vertices and q edges. G is also called a (p, q) graph. For standard terminology and notations we follow Gross and Yellen[5]. We provide a brief summary of definitions and other information which are necessary for the present investigation.

Graph labeling, where the vertices are assigned values to certain conditions have been motivated by practical problems. According to Beineke and Hegde[1] graph labeling serves as a frontier between number theory and structure of graphs. For a dynamic survey of various graph labeling problems along with extensive bibliography we refer to Gallian [2].

Definition 1.1. *A function f is called graceful labeling of a graph G if $f : V(G) \to \{0, 1, 2, \ldots, q\}$ is injective and the induced function $f^* : E(G) \to \{1, 2, \ldots, q\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective.*

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A graph which admits graceful labeling is called a *graceful graph*. Rosa[8] called such a labeling a $\beta - valuation$ and Golomb[3] subsequently called it *graceful labeling*. The famous Ringel-Kotzig tree conjecture[7] and many illustrious works on graceful graphs brought a tide of different ways of labeling the elements of graph such as odd graceful labeling, harmonious labeling etc.

Graham and Sloane[4] introduced harmonious labeling and defined as follows:

Definition 1.2. *A Graph G is said to be harmonious if there exist an injection $f : V(G) \rightarrow Z_q$ such that the induced function $f^* : E(G) \rightarrow Z_q$ defined by $f^*(uv) = (f(u) + f(v)) \, (mod \; q)$ is a bijection and f is said to be harmonious labelling of G.*

Definition 1.3. *A Graph G is said to be harmonious if there exist an injection $f : V(G) \rightarrow Z_q$ such that the induced function $f^* : E(G) \rightarrow Z_q$ defined by $f^*(uv) = (f(u) + f(v)) \, (mod \; q)$ is a bijection and f is said to be harmonious labelling of G.*

If G is a tree or has a component that is a tree, then exactly one label may be used on two vertices and the labeling function is not an injection.

Definition 1.4. *A Graph G is said to be odd harmonious if there exist an injection $f : V(G) \rightarrow \{0, 1, 2, \ldots, 2q - 1\}$ such that the induced function $f^* : E(G) \rightarrow \{1, 3, \ldots, 2q - 1\}$ defined by $f^*(uv) = f(u) + f(v)$ is an bijection. Then f is said to be odd harmonious labeling of G.*

Liang and Bai [6] have obtained the necessary conditions for the existence of odd harmonious labelling of graph. Also they proved that if G is an odd harmonious graph, then G is a bipartite graph and if $(p, q)-$ graph G is odd harmonious, then $\sqrt{q} \leq p \leq 2q - 1$. In the same paper they have proved that a cycle C_n is odd-harmonious if and only if $n \equiv 0(mod \; 4)$. A complete graph K_n is odd harmonious if and only if $n = 2$. A complete k-partite graph $K_{(n_1, n_2, \ldots, n_k)}$ is odd harmonious if and only if $k = 2$. A windmill graph K_n^t is odd harmonious if and only if $n = 2$. Caterpillars and lobsters are odd harmonious. In the same paper many ways to construct an odd harmonious graph are reported.

Definition 1.5. *The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G say G' and G''. Join each vertex u' in G' to the neighbours of the corresponding vertex v' in G''.*

Definition 1.6. *For a graph G the split graph is obtained by adding to each vertex v a new vertex v' such that v' is adjacent to every vertex that is adjacent to v in G. The resultant graph is denoted as $spl(G)$.*

2 Main Results

Theorem 2.1. *$D_2(P_n)$ is an odd harmonious graph.*

Proof. Consider two copies of P_n. Let v_1, v_2,..., v_n be the vertices of the first copy of P_n and v'_1, v'_2,..., v'_n be the vertices of the second copy of P_n. Let G be the graph $D_2(P_n)$. Then $|V(G)| = 2n$ and $|E(G)| = 4(n-1)$. To define $f : V(G) \to \{0, 1, 2, 3, \ldots, 8n-9\}$ we consider the following three cases.

Case (i) $n = 2$

The graphs $D_2(P_2)$ is to be dealt with separately and its odd harmonious labeling is shown in *Figure 1*.

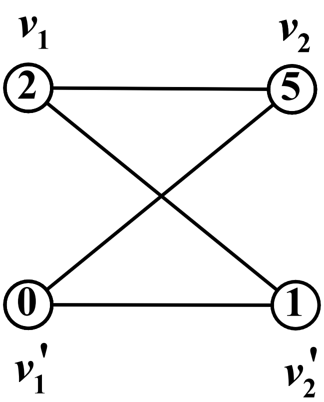

Figure 1: The graph $D_2(P_2)$ and its odd harmonious labeling

Case (ii) n is odd, $n \geq 3$

$f(v_1) = 0,$

$f(v_2) = 1,$

$f(v_{2i+1}) = 8i\,; \qquad 1 \leq i \leq \dfrac{n-1}{2}$

$f(v_{2i+2}) = 8i + 1\,; \qquad 1 \leq i \leq \dfrac{n-3}{2}$

$f(v'_1) = 4,$

$f(v'_2) = 3,$

$f(v'_{2i+1}) = 12 + 8(i-1)\,; \quad 1 \leq i \leq \dfrac{n-1}{2}$

$f(v'_{2i+2}) = 11 + 8(i-1)\,; \quad 1 \leq i \leq \dfrac{n-3}{2}$

Case (iii) n is even, $n \geq 4$

$f(v_1) = 0,$

$f(v_2) = 1,$

$f(v_{2i+1}) = 8i\ ; \qquad 1 \leq i \leq \dfrac{n-2}{2}$

$f(v_{2i+2}) = 8i + 1\ ; \qquad 1 \leq i \leq \dfrac{n-2}{2}$

$f(v_1') = 4,$

$f(v_2') = 3,$

$f(v_{2i+1}') = 12 + 8(i-1)\ ; \quad 1 \leq i \leq \dfrac{n-2}{2}$

$f(v_{2i+2}') = 11 + 8(i-1)\ ; \quad 1 \leq i \leq \dfrac{n-2}{2}$

In view of the above defined labeling pattern f is an odd harmonious labelig for $D_2(P_n)$. Hence $D_2(P_n)$ is an odd harmonious graph. ∎

Illustration 2.2. *Odd harmonious labeling of the graph $D_2(P_{11})$ is shown in Figure 2.*

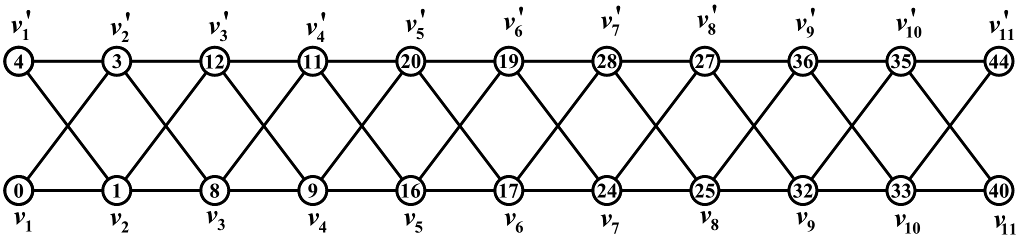

Figure 2: The graph $D_2(P_{11})$ and its odd harmonious labeling

Theorem 2.3. *$D_2(K_{1,n})$ is an odd harmonious graph.*

Proof. Consider two copies of $K_{1,n}$. Let v_1, v_2,..., v_n be the pendant vertices of the first copy of $K_{1,n}$ and v_1', v_2',..., v_n' be the pendant vertices of the second copy of $K_{1,n}$ with v are v' respective apex vertices. Let $G = D_2(K_{1,n})$. Then $|V(G)| = 2n+2$ and $|E(G)| = 4n$. We define $f : V(G) \rightarrow \{0, 1, 2, 3, \ldots, 8n-1\}$ as follows:

$f(v) = 0,$

$f(v_i) = 1 + 4(i-1)\ ; \quad 1 \leq i \leq n$

$f(v') = 2,$

$f(v_i') = f(v_n) + 4i\ ; \quad 1 \leq i \leq n$

In view of the above defined labeling pattern f is an odd harmonious labeling for

$D_2(K_{1,n})$. Hence $D_2(K_{1,n})$ is an odd harmonious graph. ∎

Illustration 2.4. *Odd harmonious labeling of the graph $D_2(K_{1,7})$ is shown in Figure 3.*

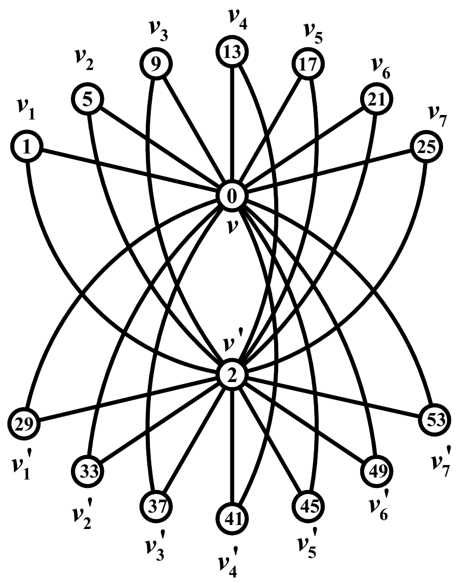

Figure 3: The graph $D_2(K_{1,7})$ and its odd harmonious labeling

Theorem 2.5. *$spl(P_n)$ is an odd harmonious graph.*

Proof. Let $u_1, u_2, u_3, \ldots, u_n$ be the vertices corresponding to $v_1, v_2, v_3, \ldots, v_n$ of P_n which are added to obtain $spl(P_n)$. Let G be the graph $spl(P_n)$. Then $|V(G)| = 2n$ and $|E(G)| = 3(n-1)$. To define $f : V(G) \to \{0, 1, 2, 3, \ldots, 6n-7\}$ we consider the following two cases.

Case (i) n is even, $n \geq 2$

$f(v_1) = 0,$

$f(v_2) = 1,$

$f(v_{2i+1}) = 6i\,; \qquad 1 \leq i \leq \dfrac{n-2}{2}$

$f(v_{2i+2}) = 6i + 1\,; \qquad 1 \leq i \leq \dfrac{n-2}{2}$

$f(u_i) = 2 + 3(i-1)$

Case (ii) n is odd, $n \geq 3$

$f(v_1) = 0,$

$f(v_2) = 1,$

$f(v_{2i+1}) = 6i\,; \quad 1 \leq i \leq \left\lceil \dfrac{n-2}{2} \right\rceil$

$f(v_{2i+2}) = 6i+1\,; \quad 1 \leq i \leq \dfrac{n-3}{2}$

$f(u_i) = 2 + 3(i-1)$

In view of the above defined labeling pattern f admits odd harmonious labelig for $spl(P_n)$. Hence $spl(P_n)$ is an odd harmonious graph. ∎

Illustration 2.6. *Odd harmonious labeling of graph* $spl(P_{10})$ *is shown in Figure 4.*

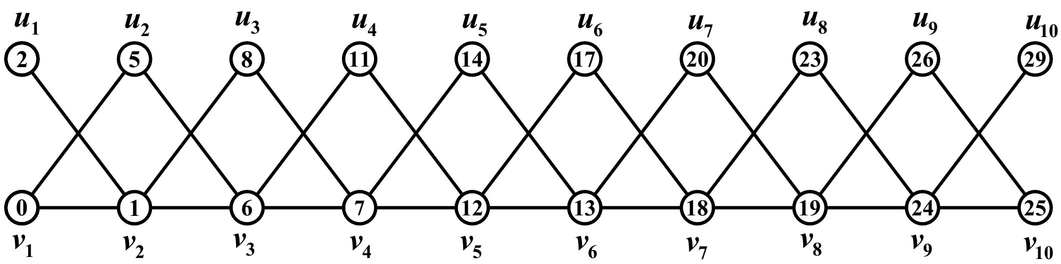

Figure 4: The graph $spl(P_{10})$ and its odd harmonious labeling

Theorem 2.7. $spl(K_{1,n})$ *is an odd harmonious graph.*

Proof. Let $v_1, v_2, v_3, \ldots, v_n$ be the pendant vertices and v be the apex vertex of $K_{1,n}$ and $u, u_1, u_2, u_3, \ldots, u_n$ are added vertices corresponding to $v, v_1, v_2, v_3, \ldots$, v_n to obtain $spl(K_{1,n})$. Let G be the graph $spl(K_{1,n})$. Then $|V(G)| = 2n+2$ and $|E(G)| = 3n$. We define vertex lebeling $f : V(G) \rightarrow \{0, 1, 2, 3, \ldots, 6n-1\}$ as follows:

$f(v) = 0$,

$f(v_i) = 2n + 1 + 4(i-1)\,; \quad 1 \leq i \leq n$

$f(u) = 2$,

$f(u_i) = 1 + 2(i-1)\,; \qquad 1 \leq i \leq n$

In view of the above defined labeling pattern f admits odd harmonious labelig for $spl(K_{1,n})$. Hence $spl(K_{1,n})$ is an odd harmonious graph. ∎

Illustration 2.8. *Odd harmonious labeling of graph* $spl(K_{1,5})$ *is shown in Figure 5.*

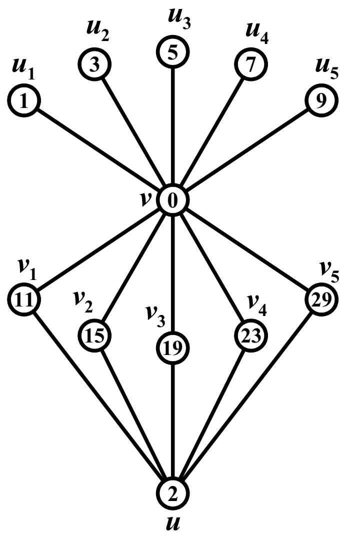

Figure 5: The graph $spl(K_{1,5})$ and its odd harmonious labeling

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