Characterization of complementary connected domination number of a graph

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Abstract
A set $S \subseteq V$ is a complementary connected dominating set if $S$ is a dominating set of $G$ and the induced subgraph $G - S$ is connected. The complementary connected domination number $\gamma_{cc}(G)$ is the minimum cardinality taken over all complementary connected dominating sets in $G$. The chromatic number is the minimum number of colours required to colour all the vertices such that adjacent vertices do not receive the same colour and is denoted by $\chi$. In this paper we characterize all cubic graphs on 8, 10, 12 vertices for which $\gamma_{cc} = \chi = 3$. 

Keywords: Complementary connected domination number, Chromatic number.

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1 Introduction

Let $G = (V, E)$ be a simple undirected graph. The degree of any vertex $u$ in $G$ is the number of edges incident with $u$ and is denoted by $d(u)$. The minimum and maximum degree of a vertex is denoted by $\delta(G)$ and $\Delta(G)$ respectively. $P_n$ denotes the path on $n$ vertices. The vertex connectivity $\kappa(G)$ of a graph $G$ is the minimum number of vertices whose removal results in a disconnected graph. A colouring of a graph is an assignment of colours to its vertices so that no two adjacent vertices have the same colour. An $n$-colouring of a graph $G$ uses $n$ colours. The chromatic number $\chi$ is defined to be the minimum $n$ for which $G$ has an $n$-colouring. If $\chi(G) = k$ but $\chi(H) < k$ for every proper subgraph $H$ of $G$, then $G$ is $k$-critical. A subset $S$ of $V$ is called a dominating set in $G$ if every vertex in $V - S$ is adjacent to at least one vertex in $S$. The minimum cardinality taken over all dominating sets in $G$ is called the domination number of $G$ and is denoted by $\gamma$.

A Set $S \subseteq V$ is a complementary connected dominating set if $S$ is a dominating set of $G$ and the induced subgraph $G - S$ is connected. The complementary connected domination number $\gamma_{cc}(G)$ is the minimum cardinality taken over all complementary connected dominating sets in $G$. The chromatic number is the minimum number of colours required to colour all the vertices such that adjacent vertices do not receive the same colour and is denoted by $\chi$.

In [16], Volkman studied graphs for which $\gamma = \beta_1$. He also investigated graphs for which $\gamma = \omega_0$ [15]. In [12] J. Paulraj Joseph and S. Arumugam investigated graphs for which $\gamma = \gamma$. In [13],
J. Paulraj Joseph analyzed graphs for which the chromatic numbers are equal to domination parameters. In [5], G. Mahadevan A. Selvam Avadyappan and A. Mydeen bibi characterized all cubic graphs on 8, 10, 12 vertices for which $\gamma_c = \chi = 3$. In [6, 7], G. Mahadevan and J. Paulraj Joseph characterized all cubic graphs on 8, 10, 12 vertices for which $\gamma = \chi = 3$. In this paper we characterize all cubic graphs on 8, 10, 12 vertices for which $\gamma_c = \chi = 3$.

**Theorem 1.1.** [10] For any graph $G$, $\gamma_c(G) \leq n - \delta$

**Theorem 1.2.** [10] For any graph $G$, $\gamma_c(G) = n - 1$ if and only if $G$ is a star. If $G$ is not a star, then $\gamma_c(G) \leq n - 2$, $(n \geq 3)$

**Theorem 1.3.** [1] For any graph $G$, $\chi(G) \leq \Delta(G) + 1$

**Theorem 1.4.** [2] If $G$ is a graph of order $p$ with maximum degree $\Delta$, then $\gamma \geq \lceil p / (\Delta + 1) \rceil$

Let $G = (V, E)$ be a connected cubic graph of order $p$ with $\gamma_c = \chi$. By Theorem 1.3, $\chi \leq 3$. Clearly, $\chi \neq 1$. We consider cubic graphs for which $\gamma_c = \chi = 3$. By Theorem 1.4, $\gamma_c \geq \lceil p / 4 \rceil$. Since $\gamma_c = 3$, $6 < p \leq 15$ and $p \neq 4$. Since $G$ is cubic we have $p$ is even and hence the possible values of $p$ are 8, 10 and 12.

2 Cubic graphs of order 8

**Theorem 2.1.** Let $G$ be a connected cubic graph on 8 vertices. Then $\gamma_c = \chi = 3$ if and only if $G$ is isomorphic to any one of the graphs given in Figure 2.1.

![Figure 2.1](image-url)

**Proof.** Let $S = \{u, v, w\}$ be a minimum complementary connected dominating set of $G$ and $V-S = \{x_1, x_2, x_3, x_4, x_5\}$. Clearly $\left< S \right> \neq K_5$. Hence we consider the following three cases.

**Case 1.** $\left< S \right> = K_3$.

Without loss of generality, let $u$ be adjacent to $x_1$, $x_2$ and $x_3$. Then $v$ is adjacent to at least one of the vertices of $N(u) = \{x_1, x_2, x_3\}$.

Subcase (a). Let $v$ be adjacent to only one vertex of $N(u)$.

Without loss of generality, let $v$ be adjacent to $x_1$. Then $v$ is adjacent to $x_4$ and $x_5$. Now $w$ is adjacent to $x_1$ or not adjacent to $x_1$. If $w$ is adjacent to $x_1$, then $w$ is adjacent to $x_2$ and $x_3$ (or equivalently $x_4$ and $x_5$), or $x_2$ (or equivalently $x_3$) and $x_4$ (or equivalently $x_5$). If $w$ is adjacent to $x_2$ and $x_3$, then $x_2$ is not adjacent to $x_4$. Hence $x_2$ must be adjacent to $x_4$ (or equivalently $x_5$) and then $x_4$ is adjacent to $x_1$ and $x_2$, which is a contradiction. Hence no such graph exists. If $w$ is adjacent to $x_1$ and $x_4$, then $x_2$ is not adjacent to $x_4$. Hence $x_2$ is adjacent to $x_3$ or $x_5$. Then in both the cases no graph exists. If $w$ is not adjacent to $x_1$. Without loss of generality, let $w$ be adjacent to $x_3, x_5$ and $x_4$. Then $x_2$ is adjacent to $x_1$ or $x_4$ or $x_5$. If $x_2$ is adjacent to $x_1$ or $x_4$ or $x_5$ then no graph exists. If $x_2$ is adjacent to $x_3$, then $x_4$ is adjacent to $x_3$ or $x_3$, or $x_4$. Then also no graph exists.
Subcase (b). Let \( v \) be adjacent to two vertices of \( N(u) \) say \( x_1 \) and \( x_2 \).

Without loss of generality, let \( v \) be adjacent to \( x_4 \). Now \( w \) is adjacent to \( x_1 \) (or equivalently \( x_2 \)) or not adjacent to \( x_1 \) (or equivalently \( x_2 \)). If \( w \) is adjacent to \( x_1 \), then \( x_3 \) is adjacent to \( x_1 \) (or equivalently \( x_2 \)) or \( w \) or \( x_5 \). Then in all the cases, \( \{u,v,w\} \) is not a complementary connected dominating set, which is a contradiction. If \( w \) is not adjacent to \( x_1 \), then without loss of generality, let \( w \) be adjacent to \( x_3 \), \( x_4 \) and \( x_5 \). Now \( x_5 \) is adjacent to \( x_3 \) and \( x_4 \) (or) \( x_1 \) and \( x_4 \) (or) \( x_1 \) and \( x_5 \) (or) \( x_2 \) and \( x_5 \). In all the cases, \( \{u,v,w\} \) is not a complementary connected dominating set, which is a contradiction.

Subcase (c). Let \( v \) be adjacent to all the vertices of \( N(u) \).

Now \( x_3 \) is adjacent to \( x_1 \) (or equivalently \( x_2 \)), or \( w \) (or equivalently \( x_4 \) or \( x_5 \)). Since \( G \) is cubic, \( x_2 \) cannot be adjacent to \( x_5 \) (or equivalently \( x_1 \)). Hence \( x_2 \) must be adjacent to \( w \). If \( x_1 \) is adjacent to \( w \) then \( w \) is adjacent to \( x_1 \) and \( x_3 \) (or) \( x_4 \) (or) \( x_1 \) and \( x_5 \). Since \( G \) is cubic, \( w \) cannot be adjacent to \( x_1 \) and \( x_3 \). Also \( w \) cannot be adjacent to \( x_1 \) and \( x_5 \). Hence \( w \) must be adjacent to \( x_4 \) and \( x_3 \). Then \( x_4 \) must be adjacent to \( x_3 \) and \( x_5 \) is adjacent to \( x_4 \). Consequently, \( \{u,v,w\} \) is not a complementary connected dominating set, which is a contradiction.

Case 2. \( <S> = K_1 \cup K_1 \).

Let \( uv \) be an edge. Without loss of generality, let \( u \) be adjacent to \( x_1 \) and \( x_2 \). Now \( w \) is adjacent to \( x_1 \), \( x_2 \) and anyone of \( \{x_3, x_4, x_5\} \) or \( w \) is adjacent to \( x_1 \) (or equivalently \( x_2 \)) and any two of \( \{x_3, x_4, x_5\} \). If \( w \) is adjacent to \( x_1 \), \( x_2 \) and \( x_3 \), then \( x_4 \) is adjacent to \( x_3 \) (or equivalently \( x_2 \)) or not adjacent to \( x_1 \) (or equivalently \( x_3 \)). If \( x_4 \) is not adjacent to \( x_1 \) (or equivalently \( x_3 \)), then \( x_4 \) is adjacent to \( x_5 \) and \( v \). Also \( x_3 \) is adjacent to \( x_1 \) and \( x_2 \) is adjacent to \( x_3 \). Hence \( G \equiv G_1 \). If \( x_4 \) is adjacent to \( x_1 \), then \( x_3 \) is adjacent to \( x_2 \) and \( x_4 \) is adjacent to \( x_3 \) and \( x_5 \). Hence \( G \equiv G_1 \). If \( x_4 \) is adjacent to \( x_1 \), then \( x_1 \) is adjacent to \( x_2 \), \( x_3 \) and \( v \) and \( x_5 \) is adjacent to \( x_4 \) and \( x_2 \) is adjacent to \( v \). Hence \( G \equiv G_1 \). If \( w \) is adjacent to \( x_1 \), \( x_2 \) and \( x_3 \), then \( x_4 \) is adjacent to \( x_1 \) or not adjacent to \( x_1 \). If \( x_5 \) is adjacent to \( x_2 \) and \( x_4 \) then \( x_3 \) is adjacent to \( x_2 \) and \( v \). Also \( x_4 \) is adjacent to \( x_1 \). Hence \( G \equiv G_1 \).

Case 3. \( <S> = P_5 \).

Let \( v \) be adjacent to \( u \) and \( w \). Without loss of generality, let \( v \) be adjacent to \( x_1 \) and \( u \) be adjacent to \( x_4 \). Now \( w \) is adjacent to \( u \), \( w \) and anyone of \( \{x_3, x_5, x_6\} \) or \( w \) is adjacent to \( x_1 \) (or equivalently \( x_2 \)) and any two of \( \{x_3, x_5, x_6\} \). If \( x_2 \) is adjacent to \( u \), \( w \) and \( x_1 \) then \( x_4 \) is adjacent to \( w \) (or equivalently \( u \)) or not adjacent to \( w \) (or equivalently \( u \)). If \( x_4 \) is adjacent to \( w \) (or equivalently \( u \)), then \( x_5 \) is adjacent to \( x_3 \) or not adjacent to \( x_4 \). If \( x_2 \) is adjacent to \( x_5 \), \( x_3 \) and \( x_4 \), then \( x_5 \) is adjacent to \( x_3 \) and \( x_4 \) is adjacent to \( x_2 \). Hence \( G \equiv G_1 \). If \( x_5 \) is not adjacent to \( x_4 \), then \( x_4 \) is adjacent to \( x_1 \), \( x_3 \), and \( x_5 \), so that \( \{u,v,w\} \) is not a complementary connected dominating set, which is a contradiction. If \( x_4 \) is not adjacent to \( w \) (or equivalently \( u \)), then \( x_4 \) is adjacent to \( x_3 \), \( x_1 \) and \( x_3 \). Also \( x_3 \) is adjacent to \( x_2 \) and \( x_5 \) is adjacent to \( x_1 \). Hence \( G \equiv G_1 \). If \( x_2 \) is adjacent to \( w \), \( x_3 \) and \( x_4 \), then \( x_5 \) is adjacent to \( w \) (or) not adjacent to \( w \). If \( x_3 \) is adjacent to \( w \), \( x_1 \), \( x_4 \), then \( x_5 \) is adjacent to \( x_4 \) and \( x_4 \). Hence \( G \equiv G_1 \). If \( x_4 \) is not adjacent to \( w \), then \( x_5 \) is adjacent to any three of \( \{x_1, u, x_5, x_4\} \). Let \( x_1 \) be adjacent to \( x_2 \), \( u \), \( x_3 \). Also \( x_4 \) is adjacent to \( x_1 \) and \( x_5 \). Hence \( G \equiv G_1 \).

3 Cubic graphs of order 10

Throughout this section, \( G \) is a connected cubic graph on 10 vertices with \( V(G) = \{u, v, w, x_1, x_2, x_3, x_4, x_5, x_6\} \). Let \( S = \{u, v, w\} \) be a minimum complementary connected dominating set. Let \( S_1 = N(u) = \{x_1, x_2, x_3\} \). Clearly \( <S> \neq K_3 \) or \( P_5 \). Hence \( <S> = K_2 \cup K_1 \) or \( \overline{K_1} \). If \( <S> = K_2 \cup K_1 \) then \( uv \) be the edge in \( <S> \). Let \( x_4 \) and \( x_5 \) be the two remaining vertices adjacent to \( v \) and also \( x_4 \) and \( x_5 \) are the remaining two vertices adjacent to \( w \). Let \( S_2 = \{x_4, x_5\} \) and \( S_3 = \{x_4, x_5\} \).
Lemma 3.1. Let $G$ be a connected cubic graph on 10 vertices with $V(G)=\{u, v, w, x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$. Let $S = \{u, v, w\}$ be a minimum complementary connected dominating set. Let $S_1 = N(u) = \{x_1, x_2, x_3\}$. If $<S> = K_2 \cup K_1$ and $<S_1> = P_3$, then $G$ is isomorphic to $G_1$ given in Figure 3.1.

![Figure 3.1](image)

Proof. Without loss of generality, let $<S_1> = P_3 = \{x_1, x_2, x_3\}$. We consider the following three cases.

Case 1. $<S_2> = <S_3> = K_2$.

Without loss of generality, let $x_1$ be adjacent to $x_7$. Since $G$ is cubic, $x_3$ is adjacent to $x_4$ and $x_5$ is adjacent to $x_6$. Hence $G \cong G_1$.

Case 2. $<S_2> = K_2$ and $<S_3> = K_2$.

Let $x_5$ be adjacent to $x_6$ and $x_7$ or $x_1$ and $x_3$ or $x_5$ (or equivalently $x_7$) and $x_1$ (or equivalently $x_6$). In all the above situation $S$ fails to be a complementary connected dominating set, which is a contradiction. Hence no graph exists.

Case 3. $<S_2> = <S_3> = K_2$.

Since $G$ is cubic, without loss of generality let $x_1$ be adjacent to $x_6$. Also $x_4$ is adjacent to $x_4$ (or equivalently $x_3$) and $x_5$ is adjacent to $x_3$ and $x_3$ and also $x_7$ is adjacent to $x_4$. Hence $G \cong G_1$.

Lemma 3.2. If $<S> = K_2 \cup K_1$ and $<S_1> = K_3$, then $G$ is isomorphic to any one of the graphs given in Figure 3.2.

![Figure 3.2](image)
Characterization of complementary connected domination number of a graph

Proof. We consider the following three cases.

Case 1. \( <S_2> = <S_1> = K_2 \).

Since \( G \) is cubic, the three vertices in \( S_1 \) are to be incident with 6 edges. But the four vertices in \( S_2 \) and \( S_1 \) can be incident with four edges. This is a contradiction. Hence no graph exists in this case.

Case 2. \( <S_2> = K_2 \) and \( <S_1> = K_2 \).

Now \( x_1 \) is adjacent to \( x_6 \) and \( x_7 \) or \( x_4 \) and \( x_5 \) or \( x_6 \) (or equivalently \( x_7 \)). If \( x_1 \) is adjacent to \( x_6 \) and \( x_7 \) then \( \{u, v, w\} \) is not a complementary connected dominating set, which is a contradiction. Hence no graph exists. If \( x_1 \) is adjacent to \( x_4 \) and \( x_6 \), then \( x_2 \) is adjacent to \( x_5 \). If \( x_2 \) is adjacent to \( x_5 \) and \( x_6 \), then \( x_3 \) is adjacent to \( x_5 \) and \( x_7 \). Hence \( G \equiv G_2 \). If \( x_1 \) is adjacent to \( x_4 \) and \( x_6 \) then \( x_2 \) is adjacent to \( x_7 \) and \( x_5 \) or \( x_4 \) and \( x_5 \). If \( x_2 \) is adjacent to \( x_5 \) and \( x_6 \), then \( x_3 \) is adjacent to \( x_5 \) and \( x_7 \). Hence \( G \equiv G_2 \). If \( x_2 \) is adjacent to \( x_4 \) and \( x_5 \), then \( x_3 \) is adjacent to \( x_3 \) and \( x_7 \). Hence \( G \equiv G_2 \).

Case 3. \( <S_2> = <S_1> = K_2 \).

\( x_1 \) is adjacent to \( x_6 \) and \( x_7 \) (or equivalently \( x_4 \) and \( x_5 \)) or \( x_4 \) (or equivalently \( x_7 \)) and \( x_4 \) (or equivalently \( x_7 \)). If \( x_1 \) is adjacent to \( x_6 \) and \( x_7 \), then \( x_2 \) is adjacent to \( x_4 \) and \( x_5 \) or \( x_6 \) (or equivalently \( x_7 \)) and \( x_4 \) (or equivalently \( x_7 \)). If \( x_2 \) is adjacent to \( x_4 \) and \( x_5 \), then \( x_3 \) is adjacent to \( x_7 \) and \( x_5 \) and also \( x_3 \) is adjacent to \( x_7 \). Hence \( G \equiv G_2 \). If \( x_1 \) is adjacent to \( x_4 \) and \( x_5 \), then \( x_2 \) is adjacent to \( x_4 \) and \( x_5 \) or \( x_6 \) and \( x_5 \) or \( x_4 \) (or equivalently \( x_7 \)) or \( x_4 \) and \( x_7 \) (or equivalently \( x_5 \)). If \( x_3 \) is adjacent to \( x_4 \) and \( x_6 \), then \( x_3 \) is adjacent to \( x_7 \) and \( x_5 \) which is a contradiction. Hence no graph exists. If \( x_2 \) is adjacent to \( x_3 \) and \( x_5 \), then \( x_3 \) is adjacent to \( x_4 \) and \( x_6 \) or \( x_3 \) and \( x_5 \) or \( x_4 \) and \( x_6 \) (or equivalently \( x_4 \) and \( x_5 \)). If \( x_3 \) is adjacent to \( x_5 \) and \( x_6 \), then \( x_3 \) is adjacent to \( x_7 \) which is a contradiction and no graph exists. If \( x_1 \) is adjacent to \( x_5 \) and \( x_5 \), then \( x_4 \) is adjacent to \( x_6 \) which is a contradiction and no graph exists. If \( x_1 \) is adjacent to \( x_5 \) and \( x_6 \), then \( x_4 \) is adjacent to \( x_5 \) and \( x_7 \) and then \( x_5 \) is adjacent to \( x_6 \) and \( x_7 \) and \( x_7 \) is adjacent to \( x_5 \) and \( x_7 \). Hence \( G \equiv G_2 \). If \( x_2 \) is adjacent to \( x_4 \) and \( x_5 \), then \( x_3 \) is adjacent to \( x_5 \) and \( x_6 \) or \( x_4 \). Hence \( G \equiv G_2 \). If \( x_2 \) is adjacent to \( x_4 \) and \( x_5 \), then \( x_4 \) is adjacent to \( x_5 \) and \( x_6 \) or \( x_5 \) and \( x_5 \) or \( x_4 \). Hence \( G \equiv G_2 \). If \( x_2 \) is adjacent to \( x_4 \) and \( x_5 \), then \( x_3 \) is adjacent to \( x_5 \) and \( x_6 \) and \( x_7 \) or \( x_5 \) and \( x_7 \). Hence \( G \equiv G_2 \). If \( x_2 \) is adjacent to \( x_5 \) and \( x_6 \), then \( x_3 \) is adjacent to \( x_5 \) and \( x_7 \). Hence \( G \equiv G_2 \). If \( x_2 \) is adjacent to \( x_5 \) and \( x_6 \), then \( x_3 \) is adjacent to \( x_5 \) and \( x_7 \). Hence \( G \equiv G_2 \). Hence no graph exists.

Lemma 3.3. If \( <S> = K_2 \cup K_1 \) and \( <S_1> = K_2 \cup K_1 \), then \( G \) is isomorphic to either \( G_3 \) given in Figure 3.1 or \( G_4 \) given in Figure 3.3.

\[ \text{Figure 3.3} \]
Proof. Let \( x_1x_2 \) be the edge in \( <S_1> \). We consider the following three cases.

Case 1. \( <S_2> = <S_1> = K_2 \).

Now \( x_1 \) is adjacent to \( x_4 \) and \( x_7 \) (or equivalently \( x_4 \) and \( x_3 \)) or \( x_4 \) and \( x_6 \) (or equivalently \( x_5 \) and \( x_7 \)). If \( x_4 \) is adjacent to \( x_4 \) and \( x_7 \), then \( x_3 \) is adjacent to \( x_4 \) (or equivalently \( x_3 \)). This implies that \( x_4 \) is adjacent to \( x_4 \), which is a contradiction. Hence no graph exists. If \( x_1 \) is adjacent to \( x_4 \) and \( x_6 \), then \( x_1 \) is adjacent to \( x_6 \) (or equivalently \( x_7 \)). Then \( x_2 \) is adjacent to \( x_7 \). Hence \( G \cong G_1 \).

Case 2. \( <S_2> = K_2 \) and \( <S_1> = \bar{K}_2 \).

Since \( G \) is cubic \( x_3 \) is adjacent to \( x_4 \) and \( x_5 \) or \( x_4 \) (or equivalently \( x_3 \)) and \( x_4 \) (or equivalently \( x_6 \)). If \( x_3 \) is adjacent to \( x_4 \) and \( x_3 \), then \( x_4 \) is adjacent to \( x_1 \) (or equivalently \( x_2 \)) or \( x_6 \) (or equivalently \( x_5 \)). If \( x_4 \) is adjacent to \( x_1 \), then \( x_3 \) is adjacent to \( x_4 \) (or equivalently \( x_5 \)) and \( x_4 \) is adjacent to \( x_2 \). Hence \( G \cong G_1 \). If \( x_4 \) is adjacent to \( x_1 \), then \( x_5 \) is adjacent to \( x_1 \) (or equivalently \( x_2 \)) and then \( x_2 \) is adjacent to \( x_1 \). Hence \( G \cong G_1 \). If \( x_3 \) is adjacent to \( x_4 \) and \( x_4 \), then \( x_3 \) is adjacent to \( x_1 \) and \( x_2 \) (or equivalently \( x_2 \)). If \( x_3 \) is adjacent to \( x_1 \) and \( x_2 \), then \( x_4 \) is adjacent to \( x_5 \) which is a contradiction. Hence no graph exists. If \( x_3 \) is adjacent to \( x_1 \) and \( x_3 \), then \( x_3 \) is adjacent to \( x_4 \). Hence \( G \cong G_1 \).

Case 3. \( <S_2> = <S_1> = \bar{K}_2 \).

Now \( x_1 \) is adjacent to \( x_1 \) and \( x_5 \) (or equivalently \( x_6 \) and \( x_7 \)) or \( x_4 \) and \( x_7 \) (or equivalently \( x_1 \) and \( x_7 \)). If \( x_2 \) is adjacent to \( x_4 \) and \( x_5 \), then \( x_2 \) is adjacent to \( x_1 \) (or equivalently \( x_2 \)) and then \( x_2 \) is adjacent to \( x_1 \). Hence \( G \cong G_1 \). If \( x_1 \) is adjacent to \( x_1 \) and \( x_5 \), then \( x_1 \) is adjacent to \( x_1 \) and \( x_6 \) and \( x_7 \). Hence \( G \cong G_1 \). If \( x_1 \) is adjacent to \( x_1 \) and \( x_6 \), then \( x_1 \) is adjacent to \( x_6 \) and \( x_7 \). Hence \( G \cong G_1 \). If \( x_1 \) is adjacent to \( x_1 \) and \( x_7 \), then \( x_1 \) is adjacent to \( x_1 \). Hence \( G \cong G_1 \). If \( x_1 \) is adjacent to \( x_1 \) and \( x_2 \), then \( x_1 \) is adjacent to \( x_1 \), which is a contradiction. Hence no graph exists.

Lemma 3.4. There is no connected cubic graph on 10 vertices with \( S \cong \bar{K}_2 \) and \( S_1 \cong P_5 \).

Proof. If \( S \cong \bar{K}_2 \), then \( v \) can be adjacent to two of the three vertices not in \( N(u) \). It is adjacent to two vertices say \( x_4 \) and \( x_6 \). Let \( S_2 = \{x_4, x_6\} \), then \( w \) be adjacent to two other vertices say \( x_5, x_7 \). Let \( S_1 = \{x_5, x_7\} \). We consider the following three cases.

Case 1. \( S \cong S_1 \cong K_2 \).

Now \( x_1 \) is adjacent to anyone of \( \{x_4, x_5, v\} \) (or equivalently any one of \( \{x_4, x_6, w\} \)). Since \( G \) is cubic, without loss of generality let \( x_1 \) be adjacent to \( v \), and \( x_3 \) be adjacent to \( w \). Then the remaining five vertices must be incident with one vertex, which is a contradiction. Hence no graph exists.

Case 2. \( S \cong K_2 \) and \( S_1 \cong \bar{K}_2 \).

Let \( x_4x_7 \) be the edge in \( S_2 \). If \( v \) is adjacent to \( x_4 \) (or equivalently \( x_3 \)), then \( v \) is adjacent to \( x_1 \) and \( x_3 \) (or \( x_4 \) and \( x_5 \)) (or \( x_4 \) (or equivalently \( x_5 \)) and \( x_7 \) (or equivalently \( x_3 \)).

Subcase (a). If \( x_4 \) is adjacent to \( x_4 \) and \( x_5 \) then \( x_4 \) is adjacent to \( x_4 \) (or equivalently \( x_3 \)) and \( w \) is adjacent to \( x_5 \), which is a contradiction. Hence no graph exists.

Subcase (b). If \( x_5 \) is adjacent to \( x_5 \) and \( x_5 \) then \( x_5 \) is adjacent to \( x_5 \) and \( w \). If \( x_5 \) is adjacent to \( x_5 \) then \( w \) is adjacent to \( x_5 \), which is a contradiction. Hence no graph exists. If \( x_5 \) is adjacent to \( w \) then \( x_5 \) is adjacent to \( x_5 \), which is a contradiction. Hence no graph exists.

Subcase (c). If \( x_5 \) is adjacent to \( x_5 \) and \( x_5 \) then \( x_5 \) is adjacent to \( x_5 \) (or equivalently \( x_3 \)). Then \( w \) is adjacent to \( x_5 \), which is a contradiction. Hence no graph exists.

Case 3. \( S \cong S_1 \cong \bar{K}_2 \).

Now \( v \) is adjacent to \( x_1 \) (or equivalently \( x_4 \)) or \( x_6 \) (or equivalently \( x_5 \)) which is impossible. Hence no graph exists.
Lemma 3.5. If \( < S > = \overline{K}_3 \) and \( < S_1 > = K_3 \cup K_1 \), then \( G \cong G_3 \) given in Figure 3.3 or \( G \cong G_1 \) given in Figure 3.1

Proof. Let \( x_1x_2 \) be the edge in \( < S_1 > \). We consider the following three cases.

Case 1. \( < S_2 > = < S_1 > = K_3 \).

\( x_1 \) is adjacent to any one of the vertices \( \{x_4, x_5, v\} \) (or equivalently any one of \( \{x_4, x_5, w\} \)) without loss of generality, let \( x_1 \) be adjacent to \( w \). Then it can be verified that no cubic graph exists satisfying the hypothesis.

Case 2. \( < S_2 > = K_3 \) and \( < S_1 > = \overline{K}_3 \).

Now \( x_3 \) is adjacent to any one of \( \{v, x_4, x_5\} \) (or \( w \) (or) \( x_4 \) (or equivalently \( x_5 \)). If \( x_1 \) is adjacent to \( v \), then \( w \) is adjacent to \( x_3 \) or \( x_4 \) (or equivalently \( x_5 \)) or \( x_2 \). If \( w \) is adjacent to \( x_1 \) then \( x_1 \) is adjacent to \( x_7 \) and then \( x_4 \) is adjacent to \( x_3 \) and \( x_4 \) and also \( x_5 \) is adjacent to \( x_1 \). Hence by Lemma 3.3, \( G \cong G_6 \). If \( w \) is adjacent to \( x_4 \), then \( x_4 \) is adjacent to \( x_2 \) and \( x_1 \) and then \( x_3 \) is adjacent to \( x_2 \) and \( x_3 \) and also \( x_3 \) is adjacent to \( x_5 \). Hence \( G \cong G_6 \), which falls under Lemma 3.3. If \( w \) is adjacent to \( x_3 \), then \( x_3 \) is adjacent to \( x_3 \) or \( x_4 \) (or equivalently \( x_1 \)). If \( x_1 \) is adjacent to \( x_3 \), then \( x_1 \) is adjacent to \( x_6 \) and \( x_5 \) which is a contradiction. Hence no graph exists. If \( x_4 \) is adjacent to \( x_6 \), then \( x_5 \) is adjacent to \( x_6 \) and \( x_5 \) which is a contradiction. Hence no graph exists. If \( x_1 \) is adjacent to \( w \), then \( x_3 \) is adjacent to \( x_3 \) and then \( x_1 \) is adjacent to \( x_3 \) and \( x_5 \) and also \( x_6 \) is adjacent to \( x_3 \) and \( v \). Hence \( G \cong G_5 \), which falls under Lemma 3.3. If \( x_1 \) is adjacent to \( x_3 \), then \( x_4 \) is adjacent to \( x_4 \) (or equivalently \( x_3 \)) and \( x_5 \). If \( x_1 \) is adjacent to \( x_4 \) then \( x_3 \) is adjacent to \( x_3 \) and \( x_5 \). Hence \( G \cong G_5 \), which falls under Lemma 3.3. If \( x_1 \) is adjacent to \( x_3 \) then \( x_1 \) is adjacent to \( x_4 \) (or equivalently \( x_3 \)) and \( x_5 \) and \( x_5 \) is adjacent to \( x_3 \). Hence \( G \cong G_1 \), as in Figure 3.1, which falls under Lemma 3.1. If \( w \) is adjacent to \( x_3 \), then \( x_1 \) and \( x_3 \) get a contradiction and hence no graph exists. If \( x_4 \) is adjacent to \( x_4 \), then \( x_3 \) is adjacent to \( x_4 \) and then \( x_1 \) is adjacent to \( x_4 \) and then \( x_1 \) is adjacent to \( w \) and \( x_3 \) is adjacent to \( v \). Hence \( G \cong G_1 \) given in Figure 3.1. If \( x_4 \) is adjacent to \( x_2 \) then also we get a contradiction and hence no graph exists.

Case 3. \( < S_2 > = < S_1 > = \overline{K}_3 \).

If \( x_1 \) is adjacent to \( v \) (or equivalently \( w \)) or \( x_4 \) (or equivalently \( x_5 \)) (or equivalently \( x_3 \)) then \( x_3 \) is adjacent to \( x_3 \). In all the cases no new graph exists.

Lemma 3.6. There is no connected cubic graph on 10 vertices with \( < S > = \overline{K}_3 \) and \( < S_1 > = \overline{K}_3 \).

Proof. We consider the following three cases.

Case 1. \( < S_2 > = < S_1 > = K_3 \).

Let \( x_1x_2 \) be the edge in \( < S_1 > \) and \( x_4x_5 \) be the edge in \( < S_1 > \). Now \( x_1 \) is adjacent to any one of \( \{v, x_4, x_5\} \) (or equivalently any two of \( \{w, x_4, x_5\} \) or any of \( \{v, w, x_4, x_5\} \) and anyone of \( \{w, x_4, x_5\} \). In all the above cases, \( \{u,v,w\} \) is not a complementary connected dominating set, which is a contradiction.

Case 2. \( < S_2 > = K_3 \) and \( < S_1 > = \overline{K}_3 \).

Let \( w \) be adjacent to any one of \( \{x_3, x_2, x_1\} \). Let \( w \) be adjacent to \( x_1 \). In this case also \( \{u,v,w\} \) is not a complementary connected dominating set, which is a contradiction.

Case 3. \( < S_2 > = < S_1 > = \overline{K}_3 \).

In this case, \( v \) is adjacent to \( x_1 \) (or equivalently \( x_3 \)) or \( x_4 \) (or equivalently \( x_5 \)). In the cases, \( \{u,v,w\} \) is not a complementary connected dominating set, which is a contradiction.

Theorem 3.7. Let \( G \) is connected cubic graph on 10 vertices. Then \( \gamma_c = \chi = 3 \) if and only if \( G \) is isomorphic to any one of graphs given in Figures 3.3, 3.2 and 3.3.

Proof. If \( G \) is any one of the graphs given in Figure 3.1, 3.2 and 3.3, then clearly \( \gamma_c = \chi = 3 \). Conversely if \( \gamma_c = \chi = 3 \), then the proof follows from the Lemmas 3.1 to 3.6.
4 Cubic graphs on 12 vertices

Let G be a connected cubic graph on 12 vertices with V(G)={u, v, w, x₁, x₂, x₃, x₄, x₅, x₆, x₇, x₈, x₉}. Let S = {u, v, w} be a complementary connected dominating set.

Let <S₁> = N(u) = {x₁, x₂, x₃}, Let <S₂> = N(v) = {x₄, x₅, x₆} and <S₃> = N(w) = {x₇, x₈, x₉}. Then we consider the following cases:

Case 1: <S₁> = <S₂> = <S₃> = P₃.

Case 2: <S₁> = <S₂> = P₃ and <S₃> = K₂∪K₁.

Case 3: <S₁> = <S₂> = P₃ and <S₃> = K₃.

Case 4: <S₁> = P₃ and <S₂> = <S₃> = K₂∪K₁.

Case 5: <S₁> = P₃ and <S₂> = K₃.

Case 6: <S₁> = <S₂> = K₂∪K₁, <S₃> = K₃.

Case 7: <S₁> = <S₂> = K₂∪K₁, <S₃> = K₃.

Case 8: <S₁> = <S₂> = <S₃> = K₂∪K₁.

Case 9: <S₁> = <S₂> = <S₃> = K₃.

Case 10: <S₁> = K₂∪K₁, <S₂> = <S₃> = K₃.

Lemma 4.1. If <S₁> = <S₂> = <S₃> = P₃ then G is isomorphic to G₁ given in Figure 4.1.

Figure 4.1

Proof. Let <S₁> = P₃ = x₁, x₂, x₃, <S₂> = P₃ = x₄, x₅, x₆ and <S₃> = P₃ = x₇, x₈, x₉. Without loss of generality, let x₁ be adjacent to x₄. Since G is cubic, x₃ must be adjacent to x₇ or x₈. Without loss of generality, let x₃ be adjacent to x₆. Then x₅ must be adjacent to x₉. Hence G ≅ G₁.

Lemma 4.2. If <S₁> = <S₂> = P₃ and <S₃> = K₂∪K₁ then G is isomorphic to the graph G₂ given in Figure 4.2.

Proof. Let <S₁> = P₃ = x₁, x₂, x₃, <S₂> = P₃ = x₄, x₅, x₆. Let x₃x₄ be the edge in <S₂>. Now x₇ is adjacent to anyone of {x₁, x₃, x₆, x₇}. Without loss of generality, let x₇ be adjacent to x₁. Then x₄ is adjacent to x₃ or anyone of {x₄, x₆}. If x₃ is adjacent to x₇ then x₇ must be adjacent to x₄ and x₆, which is a contradiction. Hence no graph exists. If x₃ is adjacent to x₄, then x₇ must be adjacent to x₅ and x₆. Hence G ≅ G₂.
Lemma 4.3. There exists no connected cubic graph on 12 vertices with \( < S_1> = < S_2> = P_3 \) and \( < S_3> = \overline{K}_3 \).

Proof. Let \( < S_1> = P_3 = x_1, x_2, x_3 \), \( < S_2> = P_3 = x_4, x_5, x_6 \). Now \( x_7 \) is adjacent to one of \( \{ x_1, x_3 \} \) and one of \( \{ x_4, x_6 \} \) (or) \( x_7 \) is adjacent to \( x_1 \) and \( x_3 \) (or equivalently \( x_4 \) and \( x_6 \)). In both the cases cubic graph does not exists.

Lemma 4.4. If \( < S_1> = P_3 \) and \( < S_2> = < S_3> = K_2 \cup K_1 \), then \( G \) is isomorphic to the graph \( G_2 \) given in Figure 4.2.

Proof. Let \( < S_1> = x_1, x_2, x_3 \). Let \( x_4x_5 \) be the edge in \( < S_2> \) and \( x_7x_8 \) be the edge in \( < S_3> \). We consider the following two cases.

Case 1. Let \( x_1 \) be adjacent to any one of \( \{ x_4, x_5, x_7, x_8 \} \). Without loss of generality, let \( x_1 \) be adjacent to \( x_4 \). Since \( G \) is cubic \( x_3 \) is adjacent to \( x_6 \) or \( x_7 \) (or equivalently \( x_8 \)). If \( x_3 \) is adjacent to \( x_6 \) \( x_5 \) is adjacent to \( x_3 \) or \( x_4 \). If \( x_5 \) is adjacent to \( x_4 \), then \( x_4 \) is adjacent to \( x_7 \). Hence no graph exists. If \( x_7 \) is adjacent to \( x_6 \), then \( x_6 \) is adjacent to \( x_4 \) and \( x_3 \) is adjacent to \( x_8 \). Hence \( G \equiv G_3 \) given in Figure 4.2. If \( x_7 \) is adjacent to \( x_3 \) (or equivalently \( x_8 \)) then since \( G \) is cubic \( x_4 \) must be adjacent to \( x_6 \) and \( x_5 \) is adjacent to \( x_4 \) and \( x_6 \). Hence \( G \equiv G_2 \) given in Figure 4.2.

Case 2. Let \( x_1 \) be adjacent to any one of \( \{ x_4, x_5 \} \). Without loss of generality, let \( x_1 \) be adjacent to \( x_4 \). Now \( x_5 \) is adjacent to \( x_3 \) or \( x_7 \) (or equivalently \( x_8 \)) or \( x_9 \). Since \( G \) is cubic \( x_2 \) is not adjacent to \( x_3 \). If \( x_4 \) is adjacent to \( x_5 \), then \( x_5 \) is adjacent to \( x_8 \) or \( x_9 \). If \( x_4 \) is adjacent to \( x_8 \), then \( x_8 \) is adjacent to \( x_2 \) and \( x_5 \), which is a contradiction. Hence no graph exists. If \( x_7 \) is adjacent to \( x_5 \), then \( x_5 \) is adjacent to \( x_6 \). If \( x_7 \) is adjacent to \( x_3 \), then \( x_6 \) is adjacent to \( x_8 \). Hence \( G \equiv G_2 \) given in Figure 4.2. If \( x_7 \) is adjacent to \( x_3 \), then \( x_5 \) is adjacent to \( x_6 \) and \( x_6 \) is adjacent to \( x_8 \). Hence \( G \equiv G_2 \) given in Figure 4.2.

Lemma 4.5. If \( < S_1> = P_3 \) and \( < S_2> = \overline{K}_3 \) and \( < S_3> = K_2 \cup K_1 \), then \( G \) is isomorphic to \( G_2 \) given in Figure 4.2.
Proof. If \( < S_1^\rightarrow = x_1x_2x_3 \). Let \( x_3 \) be the edge in \( < S_1^\rightarrow \). Since \( G \) is cubic, \( x_1 \) must be adjacent to any one of \{\( x_4, x_5, x_6 \}\}. Without loss of generality let \( x_1 \) be adjacent to \( x_4 \). We consider the following three cases.

Case 1. \( x_4 \) is adjacent to \( x_5 \). In this case, \( x_5 \) is adjacent to \( x_3 \) (or equivalently \( x_6 \)) and \( x_4 \) and then \( x_6 \) is adjacent to \( x_3 \) and \( x_5 \) which is a contradiction. Hence no graph exists.

Case 2. \( x_4 \) is adjacent to \( x_7 \) (or equivalently \( x_8 \)). In this case, \( x_8 \) is adjacent to \( x_1 \) and \( x_2 \) (or equivalently \( x_3 \)) \( x_9 \) is adjacent to \( x_1 \) and \( x_2 \). Since \( G \) is cubic, \( x_9 \) cannot be adjacent to both \( x_1 \) and \( x_2 \). If \( x_9 \) is adjacent to \( x_3 \) and \( x_4 \), then \( x_3 \) is adjacent to \( x_5 \), \( x_6 \), and \( x_7 \). If \( x_3 \) is adjacent to \( x_5 \), then \( x_4 \) is adjacent to \( x_7 \). Hence \( G \equiv G_2 \) given in Figure 4.2. If \( x_3 \) is adjacent to \( x_4 \), then \( x_3 \) is adjacent to \( x_6 \). Hence \( G \equiv G_2 \) given in Figure 4.2.

Case 3. \( x_4 \) is adjacent to \( x_3 \). Since \( G \) is cubic, \( x_3 \) must be adjacent to any one of \{\( x_5, x_6 \}\}. Let \( x_1 \) be adjacent to \( x_5 \). Now \( x_5 \) is adjacent to both \( x_3 \) and \( x_4 \) (or \( x_6 \)) is adjacent to one of \{\( x_5, x_6 \)\} and \( x_3 \). If \( x_6 \) is adjacent to \( x_3 \) and \( x_4 \), then \( x_3 \) is adjacent to \( x_6 \), which is a contradiction. Hence no graph exists. If \( x_4 \) is adjacent to \( x_3 \) and \( x_6 \), then \( x_3 \) is adjacent to \( x_6 \). Hence, \( G \equiv G_2 \) given in Figure 4.2.

Lemma 4.6. If \( < S_1^\rightarrow = P_3 \) and \( < S_2^\rightarrow = K_5 \), then \( G \) is isomorphic to \( G_2 \) given in Figure 4.2.

Proof. Let \( < S_1^\rightarrow = x_1 x_2 x_3 \). Now \( x_1 \) is adjacent to anyone of \{\( x_4, x_5, x_6, x_7, x_8, x_9 \}\}, without loss of generality, let \( x_1 \) be adjacent to \( x_4 \). Since \( G \) is cubic, \( x_4 \) is adjacent to anyone of \{\( x_5, x_6, x_7 \)\}. Now without loss of generality, let \( x_4 \) be adjacent to \( x_5 \). Then \( x_3 \) is adjacent to \( x_5 \) or anyone of \{\( x_5, x_6 \)\}. If \( x_7 \) is adjacent to \( x_5 \), then \( x_3 \) is adjacent to \( x_4 \) and \( x_6 \). Then \( x_3 \) is adjacent to \( x_4 \) and \( x_6 \) which is a contradiction. Hence no graph exists. If \( x_5 \) is adjacent to \( x_3 \) (or equivalently \( x_6 \)), then \( x_3 \) is adjacent to \( x_8 \) (or equivalently \( x_9 \)). If \( x_3 \) is adjacent to \( x_5 \) and \( x_8 \), then \( x_3 \) is adjacent to \( x_6 \). Hence \( G \equiv G_2 \) given in Figure 4.2.

Lemma 4.7. If \( < S_1^\rightarrow = < S_2^\rightarrow = K_5 \cup K_1 \), and \( < S_1^\leftarrow = K_5 \) \( G \) is isomorphic to \( G_2 \) given in Figure 4.2.

Proof. Let \( x_2 x_3 \) be the edge in \( < S_1^\rightarrow \) and \( x_4 x_5 \) be the edge in \( < S_2^\rightarrow \). Then \( x_1 \) is adjacent to any one of \{\( x_4, x_5 \)\} (or) \( x_4 \) (or) any one of \( \{x_5, x_6, x_7\} \). In all the cases, it can be verified that \( G \) is isomorphic to \( G_2 \) given in Figure 4.2.

Lemma 4.8. If \( < S_1^\rightarrow = < S_2^\rightarrow = K_5 \cup K_1 \), then \( G \) is isomorphic to \( G_2 \) given in Figure 4.2.

Proof. Let \( x_2 x_3 \) be the edge in \( < S_1^\rightarrow \) and \( x_4 x_5 \) be the edge in \( < S_2^\rightarrow \). Then \( x_1 \) is adjacent to anyone of \{\( x_4, x_5, x_7, x_8, x_9 \)\} (or) \( x_4 \) (or) any one of \( \{x_5, x_6, x_7\} \). In all the cases, it can be verified that \( G \) is isomorphic to \( G_2 \) given in Figure 4.2.

Lemma 4.9. If \( < S_1^\rightarrow = < S_2^\rightarrow = K_5 \), then \( G \) is isomorphic to \( G_2 \) given in Figure 4.2.

Proof. In this case, we consider the following two cases, \( x_1 \) is adjacent to any two of \{\( x_4, x_5, x_6 \)\} (or equivalently any two of \{\( x_7, x_8, x_9 \)\}). \( x_1 \) is adjacent to any one of \{\( x_4, x_5, x_6 \)\} and any one of \{\( x_7, x_8, x_9 \)\}. In both cases, it can be verified that \( G \) is isomorphic to \( G_2 \) given in Figure 4.2.

Lemma 4.10. If \( < S_1^\rightarrow = K_5 \cup K_1 \), \( < S_2^\rightarrow = K_5 \), then \( G \) is isomorphic to the graph \( G_2 \) given in Figure 4.2.

Proof. Since \( G \) is cubic, \( x_1 \) is adjacent to any one of \{\( x_4, x_5, x_6, x_7, x_8, x_9 \)\}. Without loss of generality, let \( x_1 \) be adjacent to \( x_4 \). Then \( x_2 \) is adjacent to \( x_4 \) (or) \( x_3 \) is adjacent to one of \{\( x_5, x_6 \)\} (or) \( x_5 \) is adjacent to anyone of \{\( x_7, x_8, x_9 \)\}.

In all the cases, it can be verified that \( G \) is isomorphic to \( G_2 \) given in Figure 4.2.

Theorem 4.11. Let \( G = (V, E) \) be a connected cubic graph on 12 vertices. Then \( G \) is isomorphic to any one of the graphs given in Figures 4.1 and 4.2 for which \( \gamma_c = \chi = 3 \).

Proof. If \( G \) is any one of the graphs given in Figure 4.1 and 4.2, then clearly \( \gamma_c = \chi = 3 \). Conversely, if \( \gamma_c = \chi = 3 \), then the proof follows from the Lemmas 4.1 to 4.10.

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References


