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On edge irregularity strength of subdivision of star S_n

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Abstract

In this paper, we study the total edge irregularity strength of some well known graphs. An edge irregular total k-labeling $\varphi : V \cup E \to \{1, 2, \ldots, k\}$ of a graph $G = (V, E)$ is a labeling of vertices and edges of G in such a way that for any different edges xy and $x'y'$ their weights are distinct. The total edge irregularity strength $tes(G)$ is defined as the minimum k for which G has an edge irregular total k-labeling. Also, we determine the exact value of the total edge irregularity strength of subdivision of star S_n.

Keywords: Irregularity strength, Total edge irregularity strength, Edge irregular total labeling, Subdivision of star.
AMS Subject Classification(2010): 05C78.

1 Introduction

Bača, Jendroľ, Miller and Ryan [3] defined the notion of an *edge irregular total k-labeling* of a graph $G = (V, E)$ to be a labeling of the vertices and edges of G, $\varphi : V \cup E \to \{1, 2, \ldots, k\}$ such that, the *edge weights* $wt_\varphi(uv) = \varphi(u) + \varphi(uv) + \varphi(v)$ are different for all edges. The minimum k for which the graph G has an edge irregular total k-labeling is called *the total edge irregularity strength of* G, $tes(G)$.

The motivation for the definition of the total edge irregularity strength came from irregular assignments and the irregularity strength of graphs introduced by Chartrand, Jacobson, Lehel, Oellermann, Ruiz and Saba [6]. An *irregular assignment* is a k-labeling of the edges $\phi : E \to \{1, 2, \ldots, k\}$ such that the sum of the labels of edges incident with a vertex is different for all the vertices of G and the smallest k for which there is an irregular assignments is the *irregularity strength*and is denoted by $s(G)$.

Finding the irregularity strength of a graph seems to be hard even for graphs with simple structure, see [4, 7, 9, 14]. Karoński, Luczak and Thomason [12] conjectured that the edges of every connected graph of order at least 3 can be assigned labels from $\{1, 2, 3\}$ such that for all the pairs of adjacent vertices, the sums of the labels of the incident edges are different.

We mention the following result from [3] giving a lower bound on the total edge irregularity strength of a graph:

$$tes(G) \geq \max\left\{ \left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil \right\}, \tag{1}$$

where $\Delta(G)$ is the maximum degree of G. The exact values of the total edge irregularity strength for paths, cycles, stars, wheels and friendship graphs are determined in [3].

Recently Ivančo and Jendroľ [8] posed the following conjecture:

Conjecture 1.1. [8] *Let G be an arbitrary graph different from K_5. Then*

$$tes(G) = \max\left\{\left\lceil\frac{|E(G)|+2}{3}\right\rceil, \left\lceil\frac{\Delta(G)+1}{2}\right\rceil\right\}. \tag{2}$$

Conjecture 1 has been verified for trees in [8], for complete graphs and complete bipartite graphs in [10] and [11], for the Cartesian product of two paths $P_n \square P_m$ in [13], for corona product of a path with certain graphs in [15], for large dense graphs with $\frac{|E(G)|+2}{3} \leq \frac{\Delta(G)+1}{2}$ in [5], for the categorical product of two paths $P_n \times P_m$ in [2] and for the categorical product of a cycle and a path $C_n \times P_m$ in [1].
Motivated by [1],[3] and [15] we investigate the total edge irregularity strength of subdivision of a star S_n. In [16], for $m \geq 0$ and $n \geq 3$, let S_n^m be a graph obtained by inserting m vertices to every edge of a star S_n. Thus, the star S_n can be written as S_n^0. The graph S_n^m is given in Figure1.

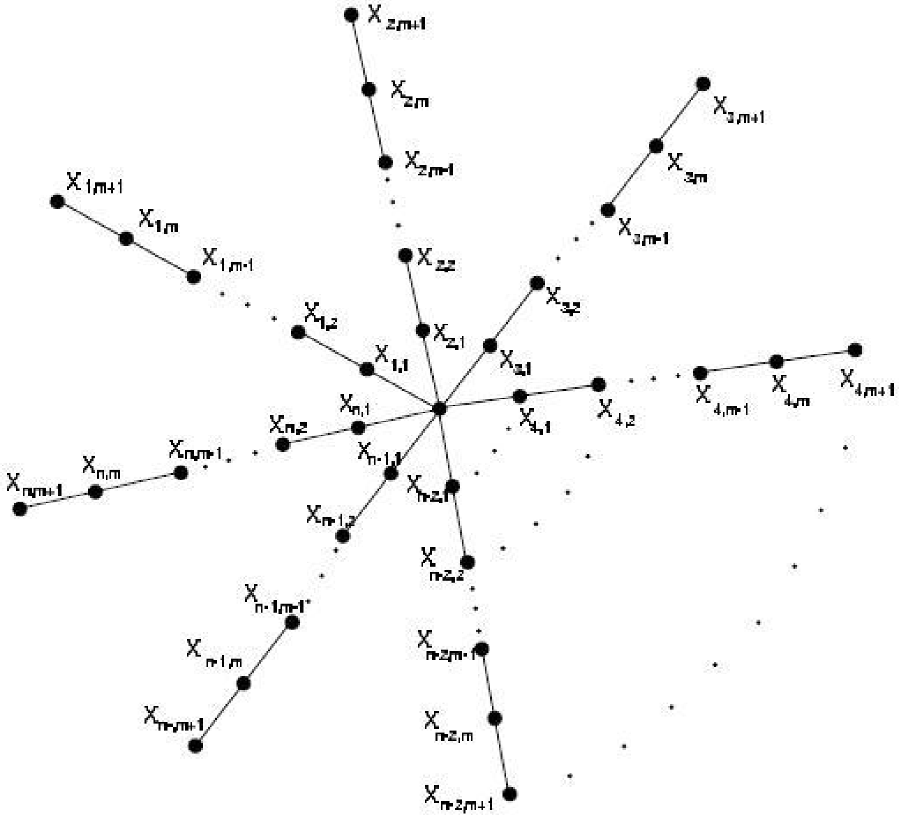

Figure 1: The graph of S_n^m

We define the vertex set and the edge set of the graph S_n^m as follows:

$$V(S_n^m) = \{c, x_{i,j} : i \in [1,n], j \in [1, m+1]\},$$

and

$$E(S_n^m) = \{cx_{i,1}, x_{i,j-1}x_{i,j} : i \in [1,n], j \in [2, m+1]\}$$

Clearly, a graph S_n^m has $mn+n+1$ vertices and $mn+n$ edges. Among these vertices, one vertex has degree n, n vertices have degree one, and the remaining vertices have degree two. As the maximum degree $\triangle(S_n^m)=n$, then (1) implies that $tes(S_n^m)\geq\left\lceil\frac{mn+n+2}{3}\right\rceil$. To show that $\left\lceil\frac{mn+n+2}{3}\right\rceil$ is an upper bound for the $tes(S_n^m)$, we describe an edge irregular total $\left\lceil\frac{mn+n+2}{3}\right\rceil$-labeling for S_n^m.

Theorem 1.2. *For* $n\geq 3$, $tes(S_n^1)=\left\lceil\frac{2n+2}{3}\right\rceil$.

Proof. The inequality $tes(S_n^1)\geq\left\lceil\frac{2n+2}{3}\right\rceil$ follows from (1). To prove $tes(S_n^1)\leq\left\lceil\frac{2n+2}{3}\right\rceil$, we split the edge set of S_n^1 in mutually disjoint subsets:

$A_i=\{cx_{i,1}\}$ for $1\leq i\leq n$ and $B_i=\{x_{i,1}x_{i,2}\}$ for $1\leq i\leq n$.

Let $k=\lceil\frac{2n+2}{3}\rceil$ and define a total k-labeling $\psi_1: V\cup E\rightarrow\{1,2,\ldots,k\}$ with $\psi_1(c)=1$ as follows:

Case 1. $when\ n\equiv 0(mod3)$

$$\psi_1(x_{i,j})=\begin{cases}1, & \text{if } 1\leq i\leq\frac{n}{3}, j=1\\ 1+i-\frac{n}{3}, & \text{if } \frac{n}{3}+1\leq i\leq n, j=1\\ k, & \text{if } 1\leq i\leq n, j=2\end{cases}$$

$$\psi_1(A_i)=\begin{cases}i, & \text{if } 1\leq i\leq\frac{n-3}{3}\\ \frac{n}{3}, & \text{if } \frac{n}{3}\leq i\leq n\end{cases}$$

$$\psi_1(B_i)=\begin{cases}\frac{n+3i}{3}, & \text{if } 1\leq i\leq\frac{n}{3}\\ k-1, & \text{if } \frac{n+3}{3}\leq i\leq n\end{cases}$$

Under the labeling ψ_1, the total weights of the edges are as follows:

(i) edges in A_i receive $2+i$ for $1\leq i\leq n$,

(ii) edges in B_i receive $\frac{3k+3+n+3i}{3}$ for $1\leq i\leq\frac{n}{3}$ and $\frac{6k+3i-n}{3}$ for $\frac{n+3}{3}\leq i\leq n$.

Case 2. $when\ n\equiv 1(mod3)$

$$\psi_1(x_{i,j})=\begin{cases}1, & \text{if } 1\leq i\leq\frac{n-1}{3}, j=1\\ 2+i-\frac{n+2}{3}, & \text{if } \frac{n+2}{3}\leq i\leq n, j=1\\ k, & \text{if } 1\leq i\leq n,\ j=2\end{cases}$$

$$\psi_1(A_i)=\begin{cases}i, & \text{if } 1\leq i\leq\frac{n-1}{3}\\ \frac{n-1}{3}, & \text{if } \frac{n+2}{3}\leq i\leq n\end{cases}$$

$$\psi_1(B_i)=\begin{cases}\frac{n+3i-1}{3}, & \text{if } 1\leq i\leq\frac{n-1}{3}\\ k-2, & \text{if } \frac{n+2}{3}\leq i\leq n\end{cases}$$

Under the labeling ψ_1, the total weights of the edges are as follows:

(i) edges in A_i receive $2+i$ for $1\leq i\leq n$,

(ii) edges in B_i receive $\frac{3k+2+n+3i}{3}$ for $1\leq i\leq\frac{n-1}{3}$ and $\frac{6k+3i-n-2}{3}$ for $\frac{n+2}{3}\leq i\leq n$

Case 3. $when\ n \equiv 2(mod3)$

$$\psi_1(x_{i,j}) = \begin{cases} 1, & \text{if } 1 \leq i \leq \frac{n+1}{3}, j = 1 \\ 2 + i - \frac{n+4}{3}, & \text{if } \frac{n+4}{3} \leq i \leq n, j = 1 \\ k, & \text{if } 1 \leq i \leq n,\ j = 2 \end{cases}$$

$$\psi_1(A_i) = \begin{cases} i, & \text{if } 1 \leq i \leq \frac{n+1}{3} \\ \frac{n+1}{3}, & \text{if } \frac{n+4}{3} \leq i \leq n \end{cases}$$

$$\psi_1(B_i) = \begin{cases} \frac{n+3i+1}{3}, & \text{if } 1 \leq i \leq \frac{n+1}{3} \\ k, & \text{if } \frac{n+4}{3} \leq i \leq n \end{cases}$$

Under the labeling ψ_1, the total weights of the edges are as follows:

(i) edges in A_i receive the integer $2 + i$ for $1 \leq i \leq n$,

(ii) edges in B_i receive the integer $\frac{3k+4+n+3i}{3}$ for $1 \leq i \leq \frac{n+1}{3}$ and $\frac{6k+3i-n+2}{3}$ for $\frac{n+4}{3} \leq i \leq n$.

It can be easily verified that the vertex and edge labels are atmost k and the edge-weights are pairwise distinct. Thus, the resulting total labeling is the desired edge irregular k-labeling. This concludes the proof. ∎

Theorem 1.3. *For* $n \geq 3$, $tes(S_n^2) = \lceil \frac{3n+2}{3} \rceil$.

Proof. The ineqality $tes(S_n^2) \geq \left\lceil \frac{3n+2}{3} \right\rceil$ follows from (1). To prove the equality we split the edge set of S_n^2 in mutually disjoint subsets:

$A_i = \{cx_{i,1}\}$, $B_i = \{x_{i,1}x_{i,2}\}$ and $C_i = \{x_{i,2}x_{i,3}\}$ for $1 \leq i \leq n$.

Let $k = \lceil \frac{3n+2}{3} \rceil$. Define a total k-labeling $\psi_2 : V \cup E \to \{1, 2, \ldots, k\}$ such that $\psi_2(c) = 1$ and for $1 \leq i \leq n$,

$$\psi_2(x_{i,j}) = \begin{cases} i, & \text{if } j = 1 \\ i + 1, & \text{if } j = 2 \\ k, & \text{if } j = 3 \end{cases}$$

$\psi_2(A_i) = 1,\ \psi_2(B_i) = k - i,\ \psi_2(C_i) = k - 1,$

Under the labeling ψ_2, the total weights of the edges are as follows:

(i) edges in A_i receive $2 + i$ for $1 \leq i \leq n$,

(ii) edges in B_i receive $k + 1 + i$ for $1 \leq i \leq n$,

(iii) edges in C_i receive $2k + i$ for $1 \leq i \leq n$.

It can be easily verified ψ_2 is an edge irregular total labeling having the required property. ∎

Theorem 1.4. *For* $n \geq 3$, $tes(S_n^3) = \lceil \frac{4n+2}{3} \rceil$.

Proof. The inequality $tes(S_n^3) \geq \left\lceil \frac{4n+2}{3} \right\rceil$ from (1). To prove the equality we split the edge set of S_n^3 in mutually disjoint subsets:

$A_i = \{cx_{i,1}\}$, $B_i = \{x_{i,1}x_{i,2}\}$, $C_i = \{x_{i,2}x_{i,3}\}$ and $D_i = \{x_{i,3}x_{i,4}\}$ for $1 \leq i \leq n$.

Let $k = \lceil \frac{4n+2}{3} \rceil$. We define a total k-labeling ψ_3 such that $\psi_3(c) = 1$ and for $1 \leq i \leq n$,

$$\psi_3(x_{i,j}) = \begin{cases} i, & \text{if } j = 1 \\ i+1, & \text{if } j = 2 \\ k, & \text{if } j = 3,4 \end{cases}$$

$\psi_3(A_i) = 1$, $\psi_3(B_i) = n + 1 - i$,

$$\psi_3(C_i) = \begin{cases} \frac{2n}{3} & \text{when } \quad n \equiv 0(mod3) \\ \frac{2n+1}{3} & \text{when } \quad n \equiv 1(mod3) \\ \frac{2n-1}{3} & \text{when } \quad n \equiv 2(mod3) \end{cases}$$

$$\psi_3(D_i) = \begin{cases} \frac{n}{3} + i & \text{when } \quad n \equiv 0(mod3) \\ \frac{5n+4}{3} - k + i & \text{when } \quad n \equiv 1(mod3) \\ \frac{5n+2}{3} - k + i & \text{when } \quad n \equiv 2(mod3) \end{cases}$$

Under the labeling ψ_3,
(i) edges in A_i receive $2 + i$ for $1 \leq i \leq n$,
(ii) edges in B_i receive $n + 2 + i$ for $1 \leq i \leq n$.

Case 1. $when\ n \equiv 0(mod3)$

(i) edges in C_i receive $k + 1 + \frac{2n}{3} + i$ for $1 \leq i \leq n$,
(ii)edges in D_i receive $2k + \frac{n}{3} + i$ for $1 \leq i \leq n$.

Case 2. $when\ n \equiv 1(mod3)$

(i) edges in C_i receive $k + 2 + \frac{2n-2}{3} + i$ for $1 \leq i \leq n$,
(ii)edges in D_i receive $k + \frac{5n+4}{3} + i$ for $1 \leq i \leq n$.

Case 3. $when\ n \equiv 2(mod3)$

(i) edges in C_i receive $k + 1 + \frac{2n-1}{3} + i$ for $1 \leq i \leq n$,
(ii)edges in D_i receive $k + \frac{5n+2}{3} + i$ for $1 \leq i \leq n$.

It can be easily verified that the vertex and edge labels are atmost k and the edge-weights are pairwise distinct.Thus, the resulting total labeling is the desired edge irregular k-labeling. ∎

Theorem 1.5. *For $4 \leq m \leq 5$ and $n \geq 3$, $tes(S_n^m) = \left\lceil \frac{(m+1)n+2}{3} \right\rceil$.*

Proof. The inequality $tes(S_n^m) \geq \left\lceil \frac{(m+1)n+2}{3} \right\rceil$. follows from (1). To prove the equality we split the edge set of S_n^m in mutually disjoint subsets:
$A_{i,1} = \{cx_{i,1}\}$ for $1 \leq i \leq n$
$A_{i,j} = \{x_{i,j-1}x_{i,j}\}$ for $1 \leq i \leq n,\ 2 \leq j \leq m+1$
Let $k = \left\lceil \frac{(m+1)n+2}{3} \right\rceil$. Define a total k-labeling ψ_4 such that $\psi_4(c) = 1$ and for $1 \leq i \leq n$,

$$\psi_4(x_{i,j}) = \begin{cases} i, & \text{if } j = 1 \\ i+1, & \text{if } j = 2 \\ n, & \text{if } j = 3 \\ k, & \text{if } j = 4,5,6 \end{cases}$$

$\psi_4(A_{i,1}) = 1$, $\psi_4(A_{i,2}) = n + 1 - i$,
$\psi_4(A_{i,3}) = n + 1$, $\psi_4(A_{i,4}) = 2n - k + 2 + i$,
$\psi_4(A_{i,5}) = 4n - 2k + 2 + i$, $\psi_4(A_{i,6}) = 5n - 2k + 2 + i$,

Under the labeling ψ_4, the edges in $A_{i,j}$ receive weights $(j-1)n + 2 + i$ for $1 \leq i \leq n$ and $1 \leq j \leq m+1$.

It is easy to verify that ψ_4 is an edge irregular total labeling having the required property. ■

Theorem 1.6. *For* $n \geq 3$, $tes(S_n^6) = \lceil \frac{7n+2}{3} \rceil$.

Proof. We have $tes(S_n^6) \geq \lceil \frac{7n+2}{3} \rceil$ from (1). To prove the equality we split the edge set of S_n^6 in mutually disjoint subsets:
$A_{i,1} = \{cx_{i,1}\}$ for $1 \leq i \leq n$
$A_{i,j} = \{x_{i,j-1}x_{i,j}\}$ for $1 \leq i \leq n,\ 2 \leq j \leq 7$.
Let $k = \lceil \frac{7n+2}{3} \rceil$. Define the total k-labeling ψ_5 such that $\psi_5(c) = 1$ and for $1 \leq i \leq n$,

$$\psi_5(x_{i,j}) = \begin{cases} i, & \text{if } j = 1 \\ i+1, & \text{if } j = 2 \\ n-1+i, & \text{if } j = 3 \\ n+i, & \text{if } j = 4 \\ k, & \text{if } j = 5, 6, 7 \end{cases}$$

$\psi_5(A_{i,1}) = 1$, $\psi_5(A_{i,2}) = n + 1 - i$,
$\psi_5(A_{i,3}) = n + 2 - i$, $\psi_5(A_{i,4}) = n + 3 - i$,
$\psi_5(A_{i,5}) = 3n - k + 2$, $\psi_5(A_{i,6}) = 5n - 2k + 2 + i$, $\psi_5(A_{i,7}) = 6n - 2k + 2 + i$,

Under the labeling ψ_5, the edges in $A_{i,j}$ receive weights $(j-1)n + 2 + i$ for $1 \leq i \leq n, 1 \leq j \leq 7$.

It can be easily verified that ψ_5 is an edge irregular total labeling having the required property. ■

Theorem 1.7. *For* $7 \leq m \leq 8$ *and* $n \geq 3$, $tes(S_n^m) = \left\lceil \frac{(m+1)n+2}{3} \right\rceil$.

Proof. The inequality $tes(S_n^m) \geq \left\lceil \frac{(m+1)n+2}{3} \right\rceil$ follows from (1). To prove the equality we split the edge set of S_n^m in mutually disjoint subsets:
$A_{i,1} = \{cx_{i,1}\}$ for $1 \leq i \leq n$
$A_{i,j} = \{x_{i,j-1}x_{i,j}\}$ for $1 \leq i \leq n,\ 2 \leq j \leq m+1$
First we construct the vertex labeling ψ_6 for $1 \leq i \leq n$ with $\psi_6(c) = 1$ and $k = \left\lceil \frac{(m+1)n+2}{3} \right\rceil$.

$$\psi_6(x_{i,j}) = \begin{cases} i, & \text{if } j = 1 \\ i+1, & \text{if } j = 2 \\ n-1+i, & \text{if } j = 3 \\ n+i, & \text{if } j = 4 \\ n+1+i, & \text{if } j = 5 \end{cases}$$

Case 1. $when\ m = 7$

$$\psi_6(x_{i,j}) = \left\{ \begin{array}{ll} k, & \text{if } 1 \le i \le n, j = 6, 7, 8 \end{array} \right.$$

Case 2. $when\ m = 8$

$$\psi_6(x_{i,j}) = \begin{cases} n + 2 + i, & \text{if } 1 \le i \le n, j = 6 \\ k, & \text{if } 1 \le i \le n, j = 7, 8, 9 \end{cases}$$

Now we define edge labeling ψ_6 as follows:

$\psi_6(A_{i,1}) = 1,\ \psi_6(A_{i,2}) = n + 1 - i,$

$\psi_6(A_{i,3}) = n + 2 - i$, $\psi_6(A_{i,4}) = n + 3 - i$, $\psi_6(A_{i,5}) = 2n + 1 - i$,

- $when\ m = 7$

  $\psi_6(A_{i,6}) = 4n - k + 1$, $\psi_6(A_{i,7}) = 6n - 2k + 2 + i$, $\psi_6(A_{i,8}) = 7n - 2k + 2 + i$,

- $when\ m = 8$

  $\psi_6(A_{i,6}) = 3n - 1 - i$, $\psi_6(A_{i,7}) = 5n - k$, $\psi_6(A_{i,8}) = 7n - 2k + 2 + i$,

  $\psi_6(A_{i,9}) = 8n - 2k + 2 + i$,

Under the labeling ψ_5, the edges in $A_{i,j}$ receive weights $(j - 1)n + 2 + i$ for $1 \le i \le n, 1 \le j \le (m + 1)$.

It can be easily verified that the vertex and edge labels are atmost k and the edge-weights of the edges from the sets $A_{i,j}$ for $i \in [1, n - 1]$, $j \in [1, m + 1]$ are pairwise distinct. Thus, the resulting total labeling is the desired edge irregular k-labeling. This concludes the proof. ∎

Open Problem. We conclude the paper with the following open problem.
For $m \ge 9$, $n \ge 3$, determine the total edge irregular strength of subdivision of star S_n.

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