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Multi-level distance labelings for generalized gear graphs

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Abstract

The *radio number* of G, $rn(G)$, is the minimum possible span. Let $d(u, v)$ denote the *distance* between two distinct vertices of a connected graph G and $diam(G)$ be the *diameter* of G. A *radio labeling* f of G is an assignment of positive integers to the vertices of G satisfying $d(u, v)+|f(u)-f(v)| \geq diam(G) + 1$. The largest integer in the range of the labeling is its span. In this paper we show that $rn(J_{t,n}) \geq \begin{cases} \frac{1}{2}(nt^2 + 2nt + 2n + 4), & \text{when t is even;} \\ \frac{1}{2}(nt^2 + 4nt + 3n + 4), & \text{when t is odd.} \end{cases}$
Further the exact value for the radio number of $J_{2,n}$ is calculated.

Keywords: multi-level distance labeling, radio number, wheel, gear, diameter.
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1 Introduction

A labeling of a graph is a map that carries graph elements to numbers (usually to the positive or non-negative integers). The most common choices of domain are the set of all vertices and edges (*total labelings*), the vertex-set alone (*vertex-labelings*), or the edge-set alone (*edge-labelings*). In this paper we consider a type of vertex labeling known as the *multi-level distance labeling* or *radio labeling* of graphs. Multi-level distance labeling is motivated by restrictions inherent in assigning channel frequencies for radio transmitters. These labelings can be considered as extensions of *distance-two labelings*.

Let $G = (V(G), E(G))$ be a simple connected graphs. The distance between the vertices u and v is denoted by $d(u, v)$ and $diam(G)$ denotes the diameter of G.

Definition 1.1. [4] *For a graph G a distance two labeling with span k is a function, $f : V(G) \rightarrow \{0, 1, ..., k\}$, such that the following are satisfied:*
1) $|f(x) - f(y)| \geq 2$ if $d(x, y) = 1$; and
2) $|f(x) - f(y)| \geq 1$ if $d(x, y) = 2$.

Definition 1.2. [1] *A radio labeling is a one-to-one mapping $f : V(G) \rightarrow \mathbb{Z}^+$ satisfying the condition*

$$d(u, v) + |f(u) - f(v)| \geq diam(G) + 1$$

for every $u, \ v \in V(G)$.

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The span of a labeling f is the maximum integer in the range of f. The radio number of G denoted by $rn(G)$ is the lowest span taken over all the radio labelings of the graph G.
The *generalized gear graph* $J_{t,n}$ is obtained from a *wheel* graph by introducing t-vertices between every pair of adjacent vertices on the cycle. The diameter of $J_{t,n}$ for $t \geq 1$ is given by

$$diam(J_{t,n}) = \begin{cases} t+2, & \text{when t is even;} \\ t+3, & \text{when t is odd.} \end{cases}$$

Theorem 1.3. [2] *The radio number of the complete graph on n vertices is n. That is, $rn(K_n) = n$.*

Lemma 1.4. [5] *For odd $n \geq 3$; $rn(P_n) = \frac{(n-1)^2}{2} + 2$ and for even $n \geq 4$, $rn(P_n) = \frac{n^2}{2} - n + 1$.*

Lemma 1.5. [1] *If G is a connected graph of order n and diameter 2, then $n \leq rn(G) \leq 2n-2$ and for every pair of integers k and n with $n \leq k \leq 2n-2$, there exists a connected graph of order n and diameter 2 with $rn(G) = k$.*

Theorem 1.6. [6] $rn(G) \geq (n-1)(diam(G)+1) - hp_{max}(DG) + 1$.

Theorem 1.7. [2] $rn(J_{1,n}) = 4n + 2$ *for* $n \geq 4$.

A complete survey on the radio number of graphs can be found in [3].

2 Lower bound for the radio number of generalized gear graph $J_{t,n}$

Theorem 2.1. $rn(J_{t,n}) \geq \begin{cases} \frac{1}{2}(nt^2 + 2nt + 2n + 4), & \text{when t is even;} \\ \frac{1}{2}(nt^2 + 4nt + 3n + 4), & \text{when t is odd.} \end{cases}$

Proof. Since there is a total of $1 + n(t+1)$ vertices in $J_{t,n}$ the total number of values required to label all the vertices of $J_{t,n}$ are $1+n(t+1)$. We count the total number of restricted values that are prohibited to be used as labels.
First of all we calculate the restricted values associated with any label of the center z. Since for any vertex $r \neq z$, $d(z,r) \leq \lceil \frac{t+2}{2} \rceil$. When t is even, the radio condition becomes

$$|f(z) - f(r)| \geq \frac{t+4}{2}.$$

Therefore, when t is even, the number of restricted values associated with any label of the center are $\frac{t+2}{2}$ (because center is assigned the minimum label).
When t is odd, then the radio condition implies that there are a total of $\frac{t+3}{2}$ which are restricted to be used as labels. Thus, the total number of restricted values associated with any label of the center, whether t is even or odd, are $\lceil \frac{t+2}{2} \rceil$.
To calculate the restricted values associated with any label of the vertices adjacent to the center let v_i denote any vertex adjacent to the center of $J_{t,n}$, then as $d(v_i, r) \leq \lceil \frac{t+4}{2} \rceil$, where $v_i \neq r$. For even t, the

radio condition becomes

$$|f(v_i) - f(r)| \geq \frac{t+2}{2}.$$

It implies that the number of restricted values associated with any label of v_i, when t is even, are t (since the restricted values above and below any label of v_i is the same).
When t is odd, the radio condition becomes

$$|f(v_i) - f(r)| \geq \frac{t+3}{2}.$$

So the number of restricted values associated with any label of v_i are $t+1$ when t is odd. But as v_n is to be assigned the maximum label, restricted values associated with any label of v_n are given by

$$\begin{cases} \frac{t}{2}, & \text{when t is even;} \\ \frac{t+1}{2}, & \text{when t is odd.} \end{cases}$$

Therefore, the total number of restricted values associated with the labels of v_i where $i = 1, 2, ..., n$ are

$$\begin{cases} (n-1)t + \frac{t}{2}, & \text{when t is even;} \\ (n-1)(t+1) + \frac{t+1}{2}, & \text{when t is odd.} \end{cases}$$

Now we calculate the restricted values associated with any label of $x_k,\ \ k \in \{\frac{t}{2}, \frac{t}{2}+1, \frac{t+1}{2}\}$, where x_k is the vertex at maximum distance between v_j and v_{j+1} on the cycle. Let $x_k,\ \ k \in \{\frac{t}{2}, \frac{t}{2}+1, \frac{t+1}{2}\}$ be the vertex at maximum distance between v_j and v_{j+1} on the cycle then as $d(x_k, r) \leq t+3,\ \ (x_k \neq r)$ where t is odd, the radio condition implies

$$|f(x_k) - f(r)| \geq 1$$

which is always true as f is a bijection. Also, as $d(x_k, r) \leq t+2,\ \ (x_k \neq r)$ when t is even, the radio condition yields

$$|f(x_k) - f(r)| \geq 1$$

. And once again the radio condition is satisfied. So there is no restricted value associated with any label of x_k.
Next, we find restricted values associated with any label of x_i, where x_i is any vertex other than x_k between v_j and v_{j+1}. If t is even then as $d(x_i, x_k) \leq \frac{t}{2} + i + 2$, where $1 \leq i \leq \frac{t}{2} - 1$, so the radio condition gives

$$|f(x_i) - f(x_k)| \geq 1 + \frac{t}{2} - i.$$

It means that the number of restricted values associated with any label of x_i are $t - 2i,\ \ 1 \leq i \leq \frac{t}{2} - 1$. Hence the total number of restricted values associated with any label of x_i where x_i is between v_j and $x_{\frac{t}{2}}$ is equal to $\sum\limits_{i=1}^{\frac{t}{2}-1} (t-2i)$———-(i)
Due to symmetry, *the number of restricted values associated with any label of* $x_i = \frac{t}{2} - i,\ \ \frac{t}{2}+2 \leq i \leq t$
Thus, the total number of restricted values associated with any label of x_i where x_i is between $x_{\frac{t}{2}+1}$

and v_{j+1} is equal to $\sum_{i=\frac{t}{2}+2}^{t}(t-2i)$————-(ii)
From (i) and (ii) the total number of restricted values associated with any label of x_i between v_j and v_{j+1} when t is even are $2\sum_{i=1}^{\frac{t}{2}-1}(t-2i)$. Therefore, the total number of restricted values associated with any label of all non-adjacent vertices to the center except $x_{\frac{t}{2}}$ and $x_{\frac{t}{2}+1}$ are $2n\sum_{i=1}^{\frac{t}{2}-1}(t-2i)=\frac{nt(t-2)}{2}$.
Now if t is odd then as $d(x_i,x_k)=\frac{t+1}{2}+i+2$, where $1\le i<\frac{t-1}{2}$, the radio condition becomes

$$|f(x_i)-f(x_k)|\ge\frac{t+3}{2}-i.$$

This implies that the number of restricted values associated with any label of x_i are $2[\frac{t+1}{2}-i]=t+1-2i,\ \ 1\le i\le\frac{t-1}{2}$. Thus, the total number of restricted values associated with any label of x_i between v_j and $x_{\frac{t+1}{2}}$ is equal to $\sum_{i=1}^{\frac{t-1}{2}}(t+1-2i)$————(iii)
Due to symmetry, we can say that the number of restricted values associated with any label of $x_i=t+1-2i,\ \ \frac{t+3}{2}\le i\le t$. It means that the total number of restricted values associated with any label of x_i between $x_{\frac{t+1}{2}}$ and v_{j+1} is equal to $\sum_{i=\frac{t+3}{2}}^{t}(t+1-2i)$————(iv)
From (iii) and (iv) the total number of restricted values associated with any label of x_i between v_j and v_{j+1} when t is odd are $2\sum_{i=1}^{\frac{t-1}{2}}(t+1-2i)$.
Therefore the total number of restricted values associated with any label of all non-adjacent vertices to the center except $x_{\frac{t+1}{2}}$ is equal to $2n\sum_{i=1}^{\frac{t-1}{2}}(t+1-2i)=\frac{n(t^2-1)}{2}$

Thus, the total number of restricted labels $=\begin{cases}\frac{1}{2}(nt^2+2), & \text{when t is even;}\\ \frac{1}{2}(nt^2+2nt+n+2), & \text{when t is odd.}\end{cases}$

Hence, $rn(J_{t,n})\ge$ Total number of values required to label all the vertices of $J_{t,n}$ + Total number of restricted values associated with any label of $J_{t,n}$.

That is, $J_{t,n}=\begin{cases}\frac{1}{2}(nt^2+2nt+2n+4), & \text{when t is even;}\\ \frac{1}{2}(nt^2+4nt+3n+4), & \text{when t is odd.}\end{cases}$ ■

Remark: Taking $t=1$ in Theorem 2.1 we get the lower bound of $J_{1,n}$ obtained in [2] and taking $t=2$ we get the lower bound of $J_{2,n}$.

Corollary 2.2. *For* $n\ge 4;\ \ rn(J_{2,n})\ge 5n+2.$

Theorem 2.3. *For* $n\ge 7, rn(J_{2,n})\le 5n+2.$

Proof. Let the center vertex of $J_{2,n}$ is labeled z, the vertices adjacent to the center are labeled sequentially $\{v_1,v_2,...,v_n\}$. The vertices not adjacent to the center are labeled sequentially $\{w_1,w_2,...,w_n\}$ and $\{u_1,u_2,...,u_n\}$, using the same orientation chosen for the v_i. If n is odd then we specify that w_1 and u_1 are adjacent to v_1 and v_2 respectively otherwise they are adjacent to v_n and v_1 respectively. The standard labelings of $J_{2,8}$ and $J_{2,9}$ are shown in Figure 3.1.

We provide a radio labeling f of $J_{2,n}$ and consequently the span of this radio labeling provides an upper bound for the radio number of $J_{2,n}$. First we define a position function that renames the vertices of $J_{2,n}$ using the set $\{x_0, x_1, ..., x_{3n}\}$, then we specify the labels $f(x_i)$ so that $i < j$ if and only if $f(x_i) < f(x_j)$. (This allows us to show more easily that f is indeed a radio labeling). Throughout this proof $n \geq 7$.

The position function $p : V(J_{2,n}) \to \{x_0, x_1, ..., x_{3n}\}$ is defined as follows.

For $n = 2k + 1$ we define

$p(z) = x_0$,

$p(w_{2i-1}) = x_i$ for $i = 1, ..., k + 1$,

$p(w_{2i}) = x_{k+1+i}$ for $i = 1, ..., k$,

$p(u_{2i-1}) = x_{2k+1+i}$ for $i = 1, ..., k + 1$,

$p(u_{2i}) = x_{2k+3+i}$ for $i = 1, ..., k$,

$p(v_i) = x_{2n+i}$ for $i = 1, ..., n$.

When $n = 2k$ the position function changes slightly in renaming the vertices w_i and u_i:

$p(z) = x_0$,

$p(w_{2i-1}) = x_i$ for $i = 1, ..., k$,

$p(w_{2i}) = x_{k+i}$ for $i = 1, ..., k$,

$p(u_{2i-1}) = x_{2k+i}$ for $i = 1, ..., k$,

$p(u_{2i}) = x_{3k+i}$ for $i = 1, ..., k$,

$p(v_i) = x_{2n+i}$ for $i = 1, ..., n$.

The above defined position function orders the vertices so that $\{x_0, x_1, ..., x_{3n}\}$ corresponds to $\{z, w_1, w_3, ..., w_n, w_2, w_4, ..., w_{n-1}, u_1, u_3, ..., u_n, u_2, u_4, ..., u_{n-1}, v_1, v_2, ..., v_n\}$ when n is odd and to $\{z, w_1, w_3, ..., w_{n-1}, w_2, w_4, ..., w_n, u_1, u_3, ..., u_{n-1}, u_2, u_4, ..., u_n, v_1, v_2, ..., v_n\}$ when n is even.

We define a labeling $f : V(J_{2,n}) \to \mathbb{Z}^+$ as follows.

$$f(x_i) = \begin{cases} 1, & i = 0; \\ 3+\text{i}, & 1 \leq i \leq 2n; \\ 2 + 2n + 3(i - 2n), & 2n + 1 \leq i \leq 3n. \end{cases}$$

Claim: The labeling f is a valid radio labeling. That is the condition

$$d(u, v) + |f(u) - f(v)| \geq 1 + diam(J_{2,n})$$

holds for all pairs of distinct vertices (u, v).

Case 1. Consider the pair (z, r) of any two distinct vertices r and z. Recall $p(z) = x_0$. As $f(x_i) \geq 5$ for any $i \geq 2$, hence the radio condition becomes $d(x_0, x_i) + |f(x_0) - f(x_i)| \geq 1 + |1 - 5| \geq 5$ for all $i \geq 2$ holds for all such vertices. This leaves the pair (z, x_1). But $p^{-1}(x_1) = w_1$, so we calculate $d(x_0, x_1) + |f(x_0) - f(x_1)| = 2 + |1 - 4| \geq 5$.

Case 2. Consider the pair (w_j, w_k) where $j \neq k$. Recall $p(w_{2i-1}) = x_i$ and note that $p(w_{2i})$ can be written as x_{n-k+i} whether n is even or odd. We have $d(w_j, w_k) = 3$ for the pairs $(w_{2i-1}, w_{2i}), (w_{2i}, w_{2i+1})$

and (w_n, w_1). These pairs are translated to (x_i, x_{n-k+i}), (x_{n-k+i}, x_{i+1}) and (x_s, x_1) respectively where $s = k+1$ when n is odd and $s = 2k$ when n is even. Now we examine the label difference for each pair:
$|f(x_i) - f(x_{n-k+i})| = n - k,$
$|f(x_{n-k+i}) - f(x_{i+1})| = n - k - 1$
and $|f(x_s) - f(x_1)|$ is k when $s = k+1$ (n odd) and is $2k-1$ when $s = 2k$ (n even).
In all the cases, using the fact that $n \geq 7$, we have that $|f(w_j) - f(w_k)| > 2$, so the radio condition is satisfied whenever $d(w_j, w_k) = 3$. Meanwhile, if j and k are not consecutive $(mod\ n)$, we have $d(w_j, w_k) = 4$. Hence the radio condition is again satisfied for all such pairs.
Case 3. The radio condition holds for all such vertices in the same way as in case 2.
Case 4. For the pair (v_j, v_k) where $i \neq j$.
As $d(v_j, v_k) = 2$. Also $|f(v_j) - f(v_k)| = |f(x_{2n+j}) - f(x_{2n+k})| \geq 3,\ \forall\ v_j, v_k$, the radio condition is satisfied.
Case 5. Consider the pair (v, w), where $v \in \{v_1, ..., v_n\}$ and $w \in \{w_1, ..., w_n\}$. We have $f(v) \in \{2n+5, 2n+8, 2n+11, ..., 5n+2\}$ and $f(w) \in \{4, 5, 6, ..., n+3\}$. For all v and w, $|f(v) - f(w)| \geq (2n+5) - (n+3) = n+2$. Now using the condition $n \geq 7$ we verify that the radio condition is satisfied for all such pairs.
Case 6. Consider the pair (v, u), where $v \in \{v_1, ..., v_n\}$ and $u \in \{u_1, ..., u_n\}$. We have $f(v) \in \{2n+5, 2n+8, 2n+11, ..., 5n+2\}$ and $f(u) \in \{n+4, n+5, n+6, ..., 2n+3\}$. For all $v \neq v_1$, $|f(v) - f(u)| \geq (2n+8) - (2n+3) = 5$. Therefore the radio condition is satisfied when $v \neq v_1$. If $v = v_1$ then as $d(v_1, u) \leq 3$, so $u = x_{n+1}$ or $u = x_{n+5}$ or $f(u) \in \{n+5, n+6, n+7, n+9, n+10, ..., 2n+3\}$. Checking the radio condition for each we obtain
$d(v_1, x_{n+1}) + |f(v_1) - f(x_{n+1})| = 1 + |(2n+5) - (n+4)| = n+2 > 5,$
$d(v_1, x_{n+5}) + |f(v_1) - f(x_{n+5})| = 2 + |(2n+5) - (n+8)| \geq n-1 > 5$ and
$d(v_1, u) + |f(v_1) - f(u)| = 3 + |(2n+5) - (2n+3)| = 5.$
Therefore once again the radio condition holds.
Case 7. Consider the pair (w, u) where $w \in \{w_1, ..., w_n\}$ and $u \in \{u_1, ..., u_n\}$.
We have $f(w) \in \{4, 5, 6, ..., n+3\}$ and $f(u) \in \{n+4, n+5, n+6, ..., 2n+3\}$.
Now $|f(w) - f(u)| = |4 - (2n+3)| = |1 - 2n| = 2n - 1$, radio Condition holds.
Also $|f(w) - f(u)| = |n+4-n-3| = 1$. But $d(w, u) = 4$. So once again condition holds.
These seven cases establish the claim that f is a radio labeling of $J_{2,n}$.
Hence $rn(J_{2,n}) \leq span(f) = f(x_{3n}) = 2 + 2n + 3(3n - 2n) = 5n + 2.$ ■

Theorem 2.4. $rn(J_{2,3}) = 21.$
Proof. Since $diam(J_{2,3}) = 4$, the radio condition becomes
$d(u, v) + |f(u) - f(v)| \geq 5$
for every two distinct vertices $u, v \in V(J_{2,3})$.

If the central vertex z has label a, then as $d(z, r) \leq 2, \forall\ r \neq z$, so by Theorem 3.1.1 there are two restricted labels associated with z.

Let x_i be any vertex on the cycle and let $f(x_1) = x$ then taking into account the vertices at maximum distance and the radio condition we see that $x+2,\ \ x+4,\ \ x+6,\ \ x+9,\ \ x+10,\ \ x+12,\ \ x+13,\ \ x+15$

and $x + 16$ are the restricted labels for all vertices.

Therefore, the allowed labels are x, $x + 1$, $x + 3$, $x + 5$, $x + 7$, $x + 8$, $x + 11$, $x + 14$ and $x + 17$. Hence, $rn(J_{2,3})$ =Total number of restricted labels + Total number of allowed labels $= 11 + 10 = 21$. ∎

Remark 2.5. *In a similar way as in Theorem 2.4, it can be shown that* $rn(J_{2,2}) = 12$, $rn(J_{2,4}) = 22$, $rn(J_{2,5}) = 28$ *and* $rn(J_{2,6}) = 32$.

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