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Super graceful labeling for some simple graphs

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Abstract

Let G be a (p, q) - graph. A bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, ..., p + q\}$ such that $f(uv) = |f(u) - f(v)|$ for every edge $uv \in E(G)$ is said to be a super graceful labeling. A graph G is called a super graceful graph if it admits a super graceful labeling. In this paper, we show that the graphs $P_n, P_m \odot nK_1, P_n^+ - e_0$, and C_n are super graceful graphs.

Keywords: Graceful labeling, Super graceful labeling and Super graceful graphs.
AMS Subject Classification(2010): 05C78.

1 Introduction

By a graph, we mean a finite undirected graph without loops or multiple edges. A path of length n is denoted by P_n. A cycle of length n is denoted by C_n. G^+ is a graph obtained from the graph G by attaching a pendent vertex to each vertex of G. The concept of graceful labeling has been introduced by Rosa [5] in 1967.

A function f is a graceful labeling of a graph G with p vertices and q edges if f is an injection from the vertices of G to the set {1,2,...,q} such that when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels are distinct. In this paper we define a new labeling called Super graceful labeling.

Let G be a (p, q) - graph. A bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, ..., p + q\}$ such that $f(uv) = |f(u) - f(v)|$ for every edge $uv \in E(G)$ is said to be a super graceful labeling. A graph G is called a super graceful graph if it admits a super graceful labeling.

Further we show that the graphs $P_n, P_m \odot nK_1, P_n^+ - e_0$ and C_n are super graceful graphs.

2 Main Results

Theorem 2.1. $P_n(n \geq 1)$, *is a super graceful graph.*

Proof. Let $P_n = v_0v_1v_2...v_n$ be a path of length n, having $n + 1$ vertices $v_0, v_1, v_2, ..., v_n$.
Now, $|V(P_n)| = n + 1, |E(P_n)| = n$.
We consider the following two cases.
Case (i). n is odd

Define $f : V(P_n) \cup E(P_n) \to \{1, 2, ..., 2n+1\}$ as follows:

$$f(v_j) = \begin{cases} 2n+1-j, & 0 \le j \le n,\ j \equiv 0\ (mod\ 2) \\ j, & 0 \le j \le n,\ j \equiv 1\ (mod\ 2) \end{cases}$$

We construct the vertex label sets as follows:

$$\begin{aligned} V_1 &= \bigcup_{\substack{j=0 \\ j\equiv 0\ (mod\ 2)}}^{n-1} \{f(v_j)\} = \bigcup_{\substack{j=0 \\ j\equiv 0\ (mod\ 2)}}^{n-1} \{2n+1-j\} \\ &= \{2n+1, 2n-1, 2n-3, ..., n+2\} \\ V_2 &= \bigcup_{\substack{j=0 \\ j\equiv 1\ (mod\ 2)}}^{n} \{f(v_j)\} = \bigcup_{\substack{j=0 \\ j\equiv 1\ (mod\ 2)}}^{n} \{j\} = \{1, 3, 5, ..., n\} \end{aligned}$$

We construct the edge label sets as follows:

$$\begin{aligned} E_1 &= \bigcup_{\substack{j=0 \\ j\equiv 0\ (mod\ 2)}}^{n-1} \{f(v_jv_{j+1})\} \\ &= \bigcup_{\substack{j=0 \\ j\equiv 0\ (mod\ 2)}}^{n-1} \{|f(v_j) - f(v_{j+1})|\} \\ &= \bigcup_{\substack{j=0 \\ j\equiv 0\ (mod\ 2)}}^{n-1} \{|(2n+1-j) - (j+1)|\} \\ &= \bigcup_{\substack{j=0 \\ j\equiv 0\ (mod\ 2)}}^{n-1} \{2(n-j)\} = \{2n, 2n-4, 2n-8, ..., 2\} \text{ and} \\ E_2 &= \bigcup_{\substack{j=0 \\ j\equiv 1\ (mod\ 2)}}^{n-2} \{f(v_jv_{j+1})\} = \bigcup_{\substack{j=0 \\ j\equiv 1\ (mod\ 2)}}^{n-2} \{|f(v_j) - f(v_{j+1})|\} \\ &= \bigcup_{\substack{j=0 \\ j\equiv 1\ (mod\ 2)}}^{n-2} \{|j - (2n+1-(j+1))|\} \\ &= \bigcup_{\substack{j=0 \\ j\equiv 1\ (mod\ 2)}}^{n-2} \{|2j-2n|\} = \bigcup_{\substack{j=0 \\ j\equiv 1\ (mod\ 2)}}^{n-2} \{2(n-j)\} \\ &= \{2n-2, 2n-6, 2n-10, ..., 4\} \end{aligned}$$

Case(ii). n is even

Define $f : V(P_n) \cup E(P_n) \to \{1, 2, ..., 2n+1\}$ as follows:

$$f(v_j) = \begin{cases} 2n+1-j, & 0 \le j \le n,\ j \equiv 0\ (mod\ 2) \\ j, & 0 \le j \le n,\ j \equiv 1\ (mod\ 2) \end{cases}$$

We construct the vertex label sets as follows:

$$
\begin{aligned}
V_1{}' &= \bigcup_{\substack{j=0\\ j\equiv 0\ (mod\ 2)}}^{n} \{f(v_j)\} = \bigcup_{\substack{j=0\\ j\equiv 0 (\mod 2)}}^{n} \{2n+1-j\}\\
&= \{2n+1, 2n-1, 2n-3, ..., n+1\}\\
V_2{}' &= \bigcup_{\substack{j=0\\ j\equiv 1\ (mod\ 2)}}^{n} \{f(v_j)\} = \bigcup_{\substack{j=0\\ j\equiv 1\ (mod\ 2)}}^{n} \{j\} = \{1, 3, 5, ..., n-1\}
\end{aligned}
$$

We construct the edge label sets as follows:

$$
\begin{aligned}
E_1^{'} &= \bigcup_{\substack{j=0\\ j\equiv 0\ (mod\ 2)}}^{n-2} \{f(v_j v_{j+1})\} = \bigcup_{\substack{j=0\\ j\equiv 0\ (mod\ 2)}}^{n-2} \{|f(v_j) - f(v_{j+1})|\}\\
&= \bigcup_{\substack{j=0\\ j\equiv 0\ (mod\ 2)}}^{n-2} \{|(2n+1-j) - (j+1)|\}\\
&= \bigcup_{\substack{j=0\\ j\equiv 0\ (mod\ 2)}}^{n-2} \{2n-2j\} = \{2n, 2n-4, 2n-8, ..., 4\}\\
E_2^{'} &= \bigcup_{\substack{j=0\\ j\equiv 1\ (mod\ 2)}}^{n-1} \{f(v_j v_{j+1})\} = \bigcup_{\substack{j=0\\ j\equiv 1\ (mod\ 2)}}^{n-1} \{|f(v_j) - f(v_{j+1})|\}\\
&= \bigcup_{\substack{j=0\\ j\equiv 1\ (mod\ 2)}}^{n-1} \{|j - (2n+1-(j+1))|\}\\
&= \bigcup_{\substack{j=0\\ j\equiv 1\ (mod\ 2)}}^{n-1} \{|2j-2n|\} = \bigcup_{\substack{j=0\\ j\equiv 1\ (mod\ 2)}}^{n-1} \{2n-2j\}\\
&= \{2n-2, 2n-6, ..., 2\}
\end{aligned}
$$

In both the cases, we observe that all the vertex label sets contain odd values and the edge label sets contain even values and hence, they are distinct and their union is $\{1, 2, ..., 2n+1\}$. Therefore, f is a super graceful labeling and hence, $P_n (n \geq 1)$ is a super graceful graph. ∎

Example 2.2. *Super graceful labelings of P_7 and P_8 are given in Figure 1 and 2 respectively.*

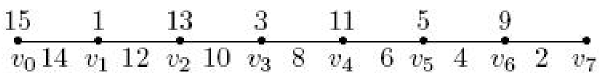

Figure 1: A super graceful labeling of P_7.

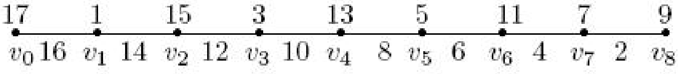

Figure 2: A super graceful labeling of P_7.

Definition 2.3. *[2], [3], [4] A connected graph having no cycle is called a tree. A tree T is called a caterpillar, if removal of all its pendent vertices results in a path. If all vertices of a path have equal number of pendent vertices, then the resulting graph is called a regular caterpillar and is denoted by* $P_m(n)$ *(or* $P_m \odot nK_1$*).*

Theorem 2.4. $P_m \odot nK_1$ *is a super graceful graph for* $m \geq 1$ *and* $n \geq 1$.

Proof. Let $v_0, v_1, v_2, ..., v_m$ be the vertices of a path P_m. Attach n pendent vertices $u_{i,1}, u_{i,2}, ..., u_{i,n}$ at each $v_i,\ 0 \leq i \leq m$. Now, $|V(P_m \odot nK_1)| = (m+1)(n+1)$ and $|E(P_m \odot nK_1)| = (m+1)(n+1)-1$
Define $f : V(P_m \odot nK_1) \cup E(P_m \odot nK_1) \to \{1, 2, ..., 2(m+1)(n+1)-1\}$ as follows:
$f(v_0) = 2(m+1)(n+1)-1$

$$f(v_i) = \begin{cases} (n+1)(2m+2-i)-1, & 0 \leq i \leq m,\ i \equiv 0\ (mod\ 2) \\ (n+1)(i+1)-1, & 0 \leq i \leq m,\ i \equiv 1\ (mod\ 2). \end{cases}$$

For $0 \leq i \leq m$ and $i \equiv 0 (mod\ 2)$,
$f(u_{i,j}) = (n+1)i + 2j - 1,\ 1 \leq j \leq n$ and
For $0 \leq i \leq m$ and $i \equiv 1\ (mod\ 2)$,
$f(u_{i,j}) = (n+1)(2m+3-i) - (2j+1),\ 1 \leq j \leq n$.
We construct the vertex label sets as follows:

$$\begin{aligned} V_1 &= \bigcup_{\substack{i=0 \\ i\equiv 0\ (mod\ 2)}}^{m} \{f(v_i)\} = \bigcup_{\substack{i=0 \\ i\equiv 0\ (mod\ 2)}}^{m} \{(n+1)((2m+2)-i)-1\} \\ &= \{(n+1)(2m+2)-1, (n+1)(2m)-1, ..., (n+1)(m+2)-1\} \\ &\quad \text{(or)} \\ &= \{(n+1)(2m+2)-1, (n+1)(2m)-1, ..., (n+1)(m+3)-1\} \\ &\quad \text{according as } m \text{ is even or odd.} \\ V_2 &= \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{m} \{f(v_i)\} = \bigcup_{\substack{i=0 \\ i\equiv 1\ (mod\ 2)}}^{m} \{(n+1)(i+1)-1\} \\ &= \{2(n+1)-1, 4(n+1)-1, 6(n+1)-1, ..., m(n+1)-1\} \\ &\quad \text{(or)} \\ &= \{2(n+1)-1, 4(n+1)-1, 6(n+1)-1, ..., (m+1)(n+1)-1\} \\ &\quad \text{according as } m \text{ is even or odd.} \\ V_3 &= \bigcup_{\substack{i=0 \\ i\equiv 0\ (mod\ 2)}}^{m} \left(\bigcup_{j=1}^{n} \{f(u_{i,j})\} \right) = \bigcup_{\substack{i=0 \\ i\equiv 0\ (mod\ 2)}}^{m} \left(\bigcup_{j=1}^{n} \{(n+1)i+2j-1\} \right) \end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{\substack{i=0\\ i\equiv 0\ (mod\ 2)}}^{m} \{(n+1)i+1,(n+1)i+3,(n+1)i+5,\cdots,(n+1)i+2n-1\}\\
&= \{1,3,5,\cdots,2n-1\}\cup\{2(n+1)+1,2(n+1)+3,2(n+1)+5,\cdots,2(n+1)\\
&\quad +2n-1\}\cup\{4(n+1)+1,4(n+1)+3,4(n+1)+5,\cdots,4(n+1)+2n-1\}\\
&\quad \cup\cdots\cup\{m(n+1)+1,m(n+1)+3,m(n+1)+5,\cdots,m(n+1)+2n-1\}\\
&\quad \text{(or)}\\
&= \{1,3,5,\cdots,2n-1\}\cup\{2(n+1)+1,2(n+1)+3,2(n+1)+5,\cdots,2(n+1)\\
&\quad +2n-1\}\cup\cdots\cup\{(m-1)(n+1)+1,(m-1)(n+1)+3,\cdots,(m-1)(n+1)\\
&\quad +2n-1\}\\
&\quad \text{according as } m \text{ is even or odd.}
\end{aligned}
$$

$$
\begin{aligned}
V_4 &= \bigcup_{\substack{i=1\\ i\equiv 1\ (mod\ 2)}}^{m}\left(\bigcup_{j=1}^{n}\{f(u_{i,j})\}\right) = \bigcup_{\substack{i=1\\ i\equiv 1\ (mod\ 2)}}^{m}\left(\bigcup_{j=1}^{n}\{(n+1)(2m+3-i)-(2j+1)\}\right)\\
&= \bigcup_{\substack{i=1\\ i\equiv 1\ (mod\ 2)}}^{m} \{(n+1)(2m+3-i)-3,(n+1)(2m+3-i)-5,\cdots,\\
&\quad (n+1)(2m+3-i)-2n-1\}\\
&= \{(n+1)(2m+2)-3,(n+1)(2m+2)-5,\cdots,(n+1)(2m+2)-2n-1\}\\
&\quad \cup\{(n+1)(2m)-3,(n+1)(2m)-5,\cdots,(n+1)(2m)-2n-1\}\cup\cdots\\
&= \cup\{(n+1)(m+4)-3,(n+1)(m+4)-5,\cdots,(n+1)(m+4)-2n-1\}\\
&\quad \text{(or)}\\
&= \{(n+1)(2m+2)-3,(n+1)(2m+2)-5,\cdots,(n+1)(2m+2)\\
&\quad -2n-1\}\cup\{(n+1)(2m)-3,(n+1)(2m)-5,\cdots,(n+1)(2m)\\
&\quad -2n-1\}\cup\cdots\cup\{(n+1)(m+3)-3,(n+1)(m+3)-5,\cdots,\\
&\quad (n+1)(m+3)-2n-1\}\\
&\quad \text{according as } m \text{ is even or odd.}
\end{aligned}
$$

We construct the edge label sets as follows:

$$
\begin{aligned}
E_1 &= \bigcup_{\substack{i=0\\ i\equiv 0\ (mod\ 2)}}^{m-1} \{f(v_iv_{i+1})\} = \bigcup_{\substack{i=0\\ i\equiv 0\ (mod\ 2)}}^{m-1} \{|f(v_i)-f(v_{i+1})|\}\\
&= \bigcup_{\substack{i=0\\ i\equiv 0\ (mod\ 2)}}^{m-1} \{|((n+1)(2m+2-i)-1)-((n+1)(i+2)-1)|\}\\
&= \bigcup_{\substack{i=0\\ i\equiv 0\ (mod\ 2)}}^{m-1} \{|(n+1)(2m+2-i-i-2)|\}
\end{aligned}
$$

$$
\begin{aligned}
&= \bigcup_{\substack{i=0\\ i\equiv 0\ (mod\ 2)}}^{m-1} \{|(n+1)(2m-2i)|\} = \bigcup_{\substack{i=0\\ i\equiv 0\ (mod\ 2)}}^{m-1} \{2(n+1)(m-i)\}\\
&= \{2(n+1)m, 2(n+1)(m-2), 2(n+1)(m-4), \cdots, 4(n+1)\}\\
&\quad \text{(or)}\\
&= \{2(n+1)m, 2(n+1)(m-2), 2(n+1)(m-4), \cdots, 2(n+1)\}\\
&\quad \text{according as } m \text{ is even or odd.}
\end{aligned}
$$

$$
\begin{aligned}
E_2 &= \bigcup_{\substack{i=1\\ i\equiv 1\ (mod\ 2)}}^{m-1} \{f(v_iv_{i+1})\} = \bigcup_{\substack{i=1\\ i\equiv 1\ (mod\ 2)}}^{m-1} \{|f(v_i) - f(v_{i+1})|\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 1\ (mod\ 2)}}^{m-1} \{|(n+1)(i+1)-1) - (((n+1)(2m+2-i-1)-1)|\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 1\ (mod\ 2)}}^{m-1} \{|(n+1)(i+1-2m-2+i+1)|\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 1\ (mod\ 2)}}^{m-1} \{|(n+1)(2i-2m)|\} = \bigcup_{\substack{i=1\\ i\equiv 1\ (mod\ 2)}}^{m-1} \{2(n+1)(m-i)\}\\
&= \{2(n+1)(m-1), 2(n+1)(m-3), \cdots, 2(n+1)\}\\
&\quad \text{(or)}\\
&= \{2(n+1)(m-1), 2(n+1)(m-3), \cdots, 4(n+1)\}\\
&\quad \text{according as } m \text{ is even or odd.}
\end{aligned}
$$

$$
\begin{aligned}
E_3 &= \bigcup_{\substack{i=0\\ i\equiv 0\ (mod\ 2)}}^{m} \left(\bigcup_{j=1}^{n} \{f(v_iu_{i,j})\}\right) = \bigcup_{\substack{i=0\\ i\equiv 0\ (mod\ 2)}}^{m} \left(\bigcup_{j=1}^{n} \{|f(v_i) - f(u_{i,j})|\}\right)\\
&= \bigcup_{\substack{i=0\\ i\equiv 0\ (mod\ 2)}}^{m} \left(\bigcup_{j=1}^{n} \{|((n+1)(2m+2-i)-i) - ((n+1)i+2j-1)|\}\right)\\
&= \bigcup_{\substack{i=0\\ i\equiv 0\ (mod\ 2)}}^{m} \left(\bigcup_{j=1}^{n} \{|(n+1)(2m+2-2i)-2j|\}\right)\\
&= \bigcup_{\substack{i=0\\ i\equiv 0\ (mod\ 2)}}^{m} \left(\bigcup_{j=1}^{n} \{2(n+1)(m+1-i)-2j\}\right)\\
&= \bigcup_{\substack{i=0\\ i\equiv 0\ (mod\ 2)}}^{m} \left(\bigcup_{j=1}^{n} \{2[(n+1)(m+1-i)-j]\}\right)
\end{aligned}
$$

$$= \bigcup_{\substack{i=0 \\ i\equiv 0 \ (mod\ 2)}}^{m} \bigcup_{j=1}^{n} \{2((n+1)(m+1-i)-1), 2((n+1)(m+1-i)-2),$$
$$\cdots, 2((n+1)(m+1-i)-n)\}$$
$$= \{2((n+1)(m+1)-1), 2((n+1)(m+1)-2), \cdots, 2((n+1)(m+1)$$
$$-n)\} \cup \{2((n+1)(m-1)-1), 2((n+1)(m-1)-2), \cdots,$$
$$2((n+1)(m-1)-n)\} \cup \cdots \cup \{2((n+1)(m-3)-1), 2((n+1)$$
$$(m-3)-2), \cdots, 2((n+1)(m-3)-n)\} \cup ... \cup \{2n, 2n-2 \cdots ., 2\}$$

(or)

$$= \{2((n+1)(m+1)-1), 2((n+1)(m+1)-2), \cdots, 2((n+1)(m+1)$$
$$-n)\} \cup \{2((n+1)(m-1)-1), 2((n+1)(m-1)-2), \cdots, 2((n+1)$$
$$(m-1)-n)\} \cup \cdots \cup \{4n+2, 4n, 4n-2, \cdots, 2n+4\}$$

according as m is even or odd.

$$E_4 = \bigcup_{\substack{i=1 \\ i\equiv 1 \ (mod\ 2)}}^{m} \left(\bigcup_{j=1}^{n} \{f(v_i u_{i,j})\}\right) = \bigcup_{\substack{i=1 \\ i\equiv 1 \ (mod\ 2)}}^{m} \left(\bigcup_{j=1}^{n} \{|f(v_i) - f(u_{i,j})|\}\right)$$
$$= \bigcup_{\substack{i=1 \\ i\equiv 1 \ (mod\ 2)}}^{m} \bigcup_{j=1}^{n} \{|((n+1)(i+1)-1) - ((n+1)2m+2-(i-1)) - 2j-1)|\}$$
$$= \bigcup_{\substack{i=1 \\ i\equiv 1 \ (mod\ 2)}}^{m} \left(\bigcup_{j=1}^{n} \{|(n+1)(i+1-2m-2+i-1)+2j|\}\right)$$
$$= \bigcup_{\substack{i=1 \\ i\equiv 1 \ (mod\ 2)}}^{m} \left(\bigcup_{j=1}^{n} \{|(n+1)(2i-2m-2)+2j|\}\right)$$
$$= \bigcup_{\substack{i=1 \\ i\equiv 1 \ (mod\ 2)}}^{m} \left(\bigcup_{j=1}^{n} \{(n+1)(2m+2-2i)-2j\}\right)$$
$$= \bigcup_{\substack{i=1 \\ i\equiv 1 \ (mod\ 2)}}^{m} \left(\bigcup_{j=1}^{n} \{2(n+1)(m+1-i)-2j\}\right)$$
$$= \bigcup_{\substack{i=1 \\ i\equiv 1 \ (mod\ 2)}}^{m} \{2((n+1)(m+1-i)-1), 2((n+1)(m+1-i)-2), \cdots,$$
$$2((n+1)(m+1-i)-n)\}$$
$$= \{2((n+1)m-1), 2((n+1)m-2), \cdots, 2((n+1)m-n)\}$$
$$\cup \{2((n+1)(m-2)-1), 2((n+1)(m-2)-2), \cdots, 2((n+1)(m-2)-n)\}$$
$$\cup \cdots \cup \{2(2(n+1)-1), 2(2(n+1)-2), \cdots, 2(2(n+1)-n)\}$$

(or)

$$
\begin{aligned}
&= \{2((n+1)m-1), 2((n+1)m-2), \cdots, 2((n+1)m-n)\} \\
&\quad \cup\{2((n+1)(m-2)-1), 2((n+1)(m-2)-2), \cdots, 2((n+1)(m-2)-n)\} \\
&\quad \cup \cdots \cup \{2((n+1)-1), 2((n+1)-2), \cdots, 2((n+1)-n)\} \\
&\quad \text{according as } m \text{ is even or odd.} \\
&= \{2((n+1)m-1), 2((n+1)m-2), \cdots, 2((n+1)m-n)\} \\
&\quad \cup\{2((n+1)(m-2)-1), 2((n+1)(m-2)-2), \cdots, 2((n+1)(m-2)-n)\} \\
&\quad \cup \cdots \cup \{4n+2, 4n, \cdots, 2n+4\} \\
&\quad \text{(or)} \\
&= \{2((n+1)m-1), 2((n+1)m-2), \cdots, 2((n+1)m-n)\} \\
&\quad \cup\{2((n+1)(m-2)-1), 2((n+1)(m-2)-2), \cdots, 2((n+1)(m-2)-n)\} \\
&\quad \cup \cdots \cup \ \{2n, 2n-2, \cdots, 2\} \\
&\quad \text{according as } m \text{ is even or odd.}
\end{aligned}
$$

We observe that all the vertex label sets are having odd values and the edge label sets are having even values and $V \cup E = \{1, 2, \cdots, 2(m+1)(n+1)-1\}$. Therefore, f is a super graceful labeling and hence $P_m \odot nK_1$ is a super graceful graph for $m \geq 1$ and $n \geq 1$. ∎

Example 2.5. *Super graceful labelings of $P_5 \odot 5K_1$ and $P_4 \odot 4K_1$ are given in Figure 3 and 4 respectively.*

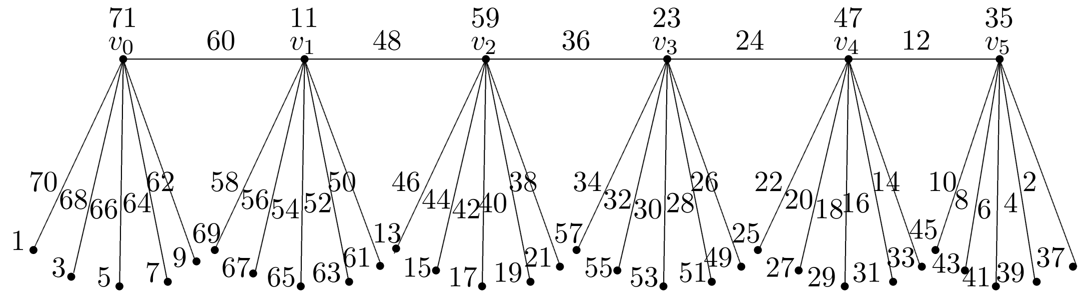

Figure 3: Super graceful labelings of $P_5 \odot 5K_1$.

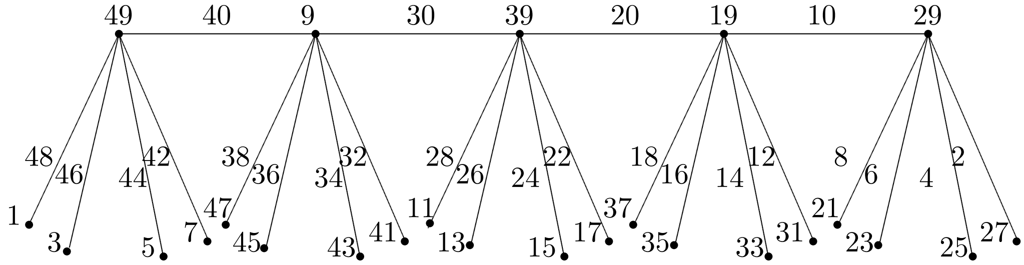

Figure 4: Super graceful labelings of $P_4 \odot 4K_1$.

Theorem 2.6. *$P_n^+ - e_0$ where $e_0 = u_0v_0$ is super graceful for $n \geq 1$.*

Proof. Let $v_0, v_1, v_2, ..., v_n$ be the vertices on the path of length n and $u_0, u_1, u_2, ..., u_n$ be the adjacent vertices to $v_0, v_1, v_2, ..., v_n$ respectively. Let $G = P_n^+ - e_0$. $|V(G)| = 2n + 1$ and $|E(G)| = 2n$. We consider the following two cases.

Case (i). n is odd.

Define $f : V(G) \cup E(G) \to \{1, 2, ..., 4n + 1\}$ as follows:

$f(u_n) = 1$ and $f(v_n) = 4n + 1$

$$f(v_i) = \begin{cases} n + 2 - i, & 0 \leq i \leq n - 1,\ i \equiv 0\ (mod\ 2) \\ 3n + 1 + i, & 0 \leq i \leq n - 1,\ i \equiv 1\ (mod\ 2) \end{cases}$$

$$f(u_i) = \begin{cases} n + 2 + i, & 1 \leq i \leq n - 1,\ i \equiv 0\ (mod\ 2) \\ 3n + 1 - i, & 1 \leq i \leq n - 1,\ i \equiv 1\ (mod\ 2) \end{cases}$$

We construct the vertex label sets as follows:

$V_1 = f(u_n) = \{1\}$ and $V_2 = f(v_n) = \{4n + 1\}$

$$\begin{aligned} V_3 &= \bigcup_{\substack{i=0 \\ i\equiv 0\ (mod\ 2)}}^{n-1} \{f(v_i)\} = \bigcup_{\substack{i=0 \\ i\equiv 0\ (mod\ 2)}}^{n-1} \{n + 2 - i\} \\ &= \{n + 2, n, n - 2, ..., 3\} \\ V_4 &= \bigcup_{\substack{i=0 \\ i\equiv 1\ (mod\ 2)}}^{n-2} \{f(v_i)\} = \bigcup_{\substack{i=0 \\ i\equiv 1\ (mod\ 2)}}^{n-2} \{3n + 1 + i\} \\ &= \{3n + 2, 3n + 4, ..., 4n - 1\} \\ V_5 &= \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-1} \{f(u_i)\} = \bigcup_{\substack{i=1 \\ i\equiv 0\ (mod\ 2)}}^{n-1} \{n + 2 + i\} \\ &= \{n + 4, n + 6, ..., 2n + 1\} \text{ and} \\ V_6 &= \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-2} \{f(u_i)\} = \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-2} \{3n + 1 - i\} \\ &= \{3n, 3n - 2, ..., 2n + 3\} \end{aligned}$$

We construct the edge label sets as follows:

$$\begin{aligned} E_1 &= \bigcup_{\substack{i=0 \\ i\equiv 0\ (mod\ 2)}}^{n-1} \{f(v_iv_{i+1})\} = \bigcup_{\substack{i=0 \\ i\equiv 0\ (mod\ 2)}}^{n-1} \{|f(v_i) - f(v_{i+1})|\} \\ &= \bigcup_{\substack{i=0 \\ i\equiv 0\ (mod\ 2)}}^{n-1} \{|(n + 2 - i) - (3n + 1 + (i + 1))|\} \\ &= \bigcup_{\substack{i=0 \\ i\equiv 0\ (mod\ 2)}}^{n-1} \{|(n + 2 - i) - (3n + 2 + i)|\} \end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{\substack{i=0\\ i\equiv 0\ (mod\ 2)}}^{n-1} \{|-2n-2i|\} = \bigcup_{\substack{i=0\\ i\equiv 0\ (mod\ 2)}}^{n-1} \{2(n+i)\}\\
&= \{2n, 2n+4, ..., 4n-2\}\\
E_2 &= \bigcup_{\substack{i=1\\ i\equiv 1\ (mod\ 2)}}^{n-2} \{f(v_iv_{i+1)}\} = \bigcup_{\substack{i=1\\ i\equiv 1\ (mod\ 2)}}^{n-2} \{|f(v_i) - f(v_{i+1})|\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 1\ (mod\ 2)}}^{n-2} \{|(3n+1+(i+1)) - (n+2-i)|\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 1\ (mod\ 2)}}^{n-2} \{|(3n+2+i) - (n+2-i)|\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 1\ (mod\ 2)}}^{n-2} \{(2n+2i)\} = \{2n+2, 2n+6, ..., 4n-4\}\\
E_3 &= \bigcup_{\substack{i=1\\ i\equiv 0\ (mod\ 2)}}^{n-1} \{f(v_iu_i)\} = \bigcup_{\substack{i=1\\ i\equiv 0\ (mod\ 2)}}^{n-1} \{|f(v_i) - f(u_i)|\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 0\ (mod\ 2)}}^{n-1} \{|(n+2-i) - (n+2+i)|\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 0\ (mod\ 2)}}^{n-1} \{2i\} = \{4, 8, 12, ..., 2n-2\}\\
E_4 &= \bigcup_{\substack{i=1\\ i\equiv 1\ (mod\ 2)}}^{n-2} \{f(v_iu_i)\} = \bigcup_{\substack{i=1\\ i\equiv 1\ (mod\ 2)}}^{n-2} \{|f(v_i) - f(u_i)|\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 1\ (mod\ 2)}}^{n-2} \{|(3n+i) - (3n-i)|\} = \bigcup_{\substack{i=1\\ i\equiv 1\ (mod\ 2)}}^{n-2} \{2i\}\\
&= \{2, 6, 10, ..., 2n-4\} \text{ and}\\
E_5 &= \{f(u_nv_n)\} = \{|f(u_n) - f(v_n)|\} = \{|1-(4n+1)|\}\\
&= \{|-4n|\} = \{4n\}
\end{aligned}
$$

Case (ii). n is even

Define $f : V(G) \cup E(G) \to \{1, 2, ..., 4n+1\}$ as follows.

$f(u_n) = 1$ and $f(v_n) = 4n+1$.

$$f(v_i) = \begin{cases} 3n+1+i, & 0 \le i \le n-1,\ i \equiv 0\ (mod\ 2) \\ n+2-i, & 0 \le i \le n-1,\ i \equiv 1\ (mod\ 2) \end{cases}$$

$$f(u_i) = \begin{cases} 3n+1-i, & 1 \le i \le n-1,\ i \equiv 0\ (mod\ 2) \\ n+2+i, & 1 \le i \le n-1,\ i \equiv 1\ (mod\ 2) \end{cases}$$

We construct the vertex label sets as follows:
$V_1^{'} = \{f(u_n)\} = 1$ and $V_2^{'} = \{f(v_n)\} = \{4n+1\}$

$$\begin{aligned}
V_3^{'} &= \bigcup_{\substack{i=0 \\ i\equiv 0\ (mod\ 2)}}^{n-2} \{f(v_i)\} = \bigcup_{\substack{i=0 \\ i\equiv 0\ (mod\ 2)}}^{n-2} \{3n+1+i\} \\
&= \{3n+1, 3n+3, ..., 4n-1\} \\
V_4^{'} &= \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-1} \{f(v_i)\} = \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-1} \{n+2-i\} \\
&= \{n+1, n-1, ..., 3\} \\
V_5^{'} &= \bigcup_{\substack{i=1 \\ i\equiv 0\ (mod\ 2)}}^{n-2} \{f(u_i)\} = \bigcup_{\substack{i=1 \\ i\equiv 0\ (mod\ 2)}}^{n-2} \{3n+1-i\} \\
&= \{3n-1, 3n-3, ..., 2n+3\} \text{ and} \\
V_6^{'} &= \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-1} \{f(u_i)\} = \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-1} \{n+2+i\} \\
&= \{n+3, n+5, ..., 2n+1\}
\end{aligned}$$

We construct the edge label sets as follows:

$$\begin{aligned}
E_1^{'} &= \bigcup_{\substack{i=0 \\ i\equiv 0\ (mod\ 2)}}^{n-2} \{f(v_i v_{i+1})\} = \bigcup_{\substack{i=0 \\ i\equiv 0\ (mod\ 2)}}^{n-2} \{|f(v_i) - f(v_{i+1})|\} \\
&= \bigcup_{\substack{i=0 \\ i\equiv 0\ (mod\ 2)}}^{n-2} \{|(3n+1+i) - (n+2-(i+1))|\} \\
&= \bigcup_{\substack{i=0 \\ i\equiv 0\ (mod\ 2)}}^{n-2} \{|(3n+1+i) - (n+1-i)|\} \\
&= \bigcup_{\substack{i=0 \\ i\equiv 0\ (mod\ 2)}}^{n-2} \{2n+2i\} = \{2n, 2n+4, ..., 4n-4\} \\
E_2^{'} &= \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-1} \{f(v_i v_{i+1})\} = \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-1} \{|f(v_i) - f(v_{i+1})|\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-1} \{|(n+2-i) - (3n+1+i+1)|\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-1} \{|-2n-2i|\} = \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-1} \{2n+2i\}
\end{aligned}$$

$$
\begin{aligned}
&= \{2n+2, 2n+6, ..., 4n-2\} \\
E_3^{'} &= \bigcup_{\substack{i=1 \\ i\equiv 0\ (mod\ 2)}}^{n-1} \{f(v_iu_i)\} = \bigcup_{\substack{i=1 \\ i\equiv 0\ (mod\ 2)}}^{n-1} \{|f(v_i)-f(u_i)|\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 0\ (mod\ 2)}}^{n-1} \{|(3n+1+i)-(3n+1-i)|\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 0\ (mod\ 2)}}^{n-1} \{|2i|\} = \{4, 8, 12, ..., 2n-4\} \\
E_4^{'} &= \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-1} \{f(v_iu_i)\} = \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-1} \{|f(v_i)-f(u_i)|\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-1} \{|(n+2-i)-(n+2+i)|\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-1} \{|-2i|\} = \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-1} \{2i\} = \{2, 6, 10, ..., 2n-2\} \\
E_5^{'} &= \{f(v_nu_n)\} = \{|f(v_n)-f(u_n)|\} = \{|(4n+1)-1|\} = \{4n\}
\end{aligned}
$$

In both the cases, we observe that all the vertex label sets are having odd values and the edge label sets are having even values and hence, they are distinct. Their union is $\{1, 2, ..., 4n+1\}$. Therefore, f is a super graceful labeling and hence, $P_n^+ - e_0$ is a super graceful graph. ■

Example 2.7. *Super graceful labelings of the graphs* $P_7^+ - e_0$ *and* $P_8^+ - e_0$ *are given in Figure 5 and 6 respectively.*

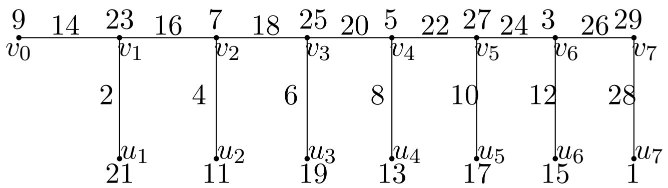

Figure 5: Super graceful labelings of the graph $P_7^+ - e_0$.

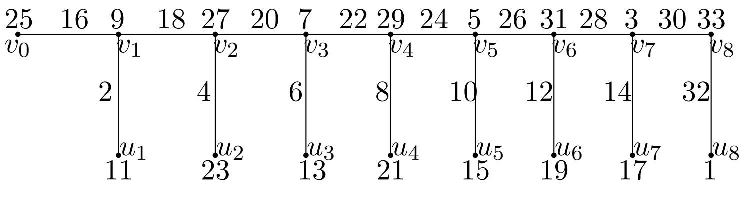

Figure 6: Super graceful labelings of the graph $P_8^+ - e_0$.

Theorem 2.8. *Any cycle $C_n (n \geq 3)$ is a super graceful graph.*

Proof. Let $V(C_n) = \{u_1, u_2, ..., u_n\}$
Now, $|V(C_n)| = n$ and $|E(C_n)| = n$.
We consider the following two cases.
Case (i). n is odd, that is, $n = 2m + 1$ for $m \geq 1$.
Define $f : V(C_n) \cup E(C_n) \rightarrow \{1, 2, ..., 4m + 2\}$ as follows:
$f(u_{2i-1}) = 2i - 1, 1 \leq i \leq m$
$f(u_{2i}) = 4m + 1 - 2i, 1 \leq i \leq m - 1$
$f(u_{2m}) = 4m \; and \; f(u_{2m+1}) = 4m + 2$
We construct the vertex label sets as follows:

$$\begin{aligned} V_1 &= \bigcup_{i=1}^{m}\{f(u_{2i-1})\} = \bigcup_{i=1}^{m}\{2i-1\} = \{1, 3, 5, ..., 2m-1\} \\ V_2 &= \bigcup_{i=1}^{m-1}\{f(u_{2i})\} = \bigcup_{i=1}^{m-1}\{4m+1-2i\} \\ &= \{4m-1, 4m-3, ..., 2m+3\} \\ V_3 &= \{f(u_{2m})\} = \{4m\} \; and \\ V_4 &= \{f(u_{2m+1})\} = \{4m+2\}. \end{aligned}$$

We construct the edge label sets as follows:

$$\begin{aligned} E_1 &= \bigcup_{i=1}^{m-1}\{f(u_{2i-1}u_{2i})\} = \bigcup_{i=1}^{m-1}\{|f(u_{2i-1}) - f(u_{2i})|\} \\ &= \bigcup_{i=1}^{m-1}\{|(2i-1) - (4m+1-2i)|\} = \bigcup_{i=1}^{m-1}\{|4i-4m-2|\} \\ &= \bigcup_{i=1}^{m-1}\{4m-4i+2\} = \{4m-2, 4m-6, 4m-10, ..., 6\} \\ E_2 &= \bigcup_{i=1}^{m-1}\{f(u_{2i}u_{2i+1})\} = \bigcup_{i=1}^{m-1}\{|f(u_{2i}) - f(u_{2i+1})|\} \\ &= \bigcup_{i=1}^{m-1}\{|(4m+1-2i) - (2i+1)|\} = \bigcup_{i=1}^{m-1}\{|4m-4i|\} \\ &= \bigcup_{i=1}^{m-1}\{4(m-i)\} = \{4(m-1), 4(m-2), ..., 4\}. \\ E_3 &= \{f(u_{2m-1}u_{2m}\} = \{|f(u_{2m-1}) - f(u_{2m})|\} \\ &= \{|(2m-1) - 4m|\} = \{2m+1\} \\ E_4 &= \{f(u_{2m}u_{2m+1})\} = \{|f(u_{2m}) - f(u_{2m+1})|\} \\ &= \{|4m - (4m+2)|\} = \{2\} \; and \\ E_5 &= \{f(u_{2m+1}u_1)\} = \{|f(u_{2m+1}) - f(u_1)|\} \end{aligned}$$

$$= \{(4m+2)-1\} = \{4m+1\}$$

Case (ii). n is even, that is, $n = 2m, m \geq 2$.
Define $f : V(C_n) \cup E(C_n) \rightarrow \{1, 2, ..., 4m\}$ as follows:
$f(u_{2i-1}) = 2i - 1, 1 \leq i \leq m-1,$
$f(u_{2i}) = 4m - 1 - 2i, 1 \leq i \leq m-1,$
$f(u_{2m-1}) = 2 \; and \; f(u_{2m}) = 4m.$
We construct the vertex label sets as follows:

$$\begin{aligned}
V_1' &= \bigcup_{i=1}^{m-1} \{f(u_{2i-1})\} = \bigcup_{i=1}^{m-1} \{2i-1\} = \{1, 3, ..., 2m-3\} \\
V_2' &= \bigcup_{i=1}^{m-1} \{f(u_{2i})\} = \bigcup_{i=1}^{m-1} \{4m-1-2i\} \\
&= \{4m-3, 4m-5, ..., 2m+1\} \\
V_3' &= \{f(u_{2m-1})\} = \{2\} \\
V_4' &= \{f(u_{2m})\} = \{4m\}
\end{aligned}$$

We construct the edge label sets as follows:

$$\begin{aligned}
E_1' &= \bigcup_{i=1}^{m-1} \{f(u_{2i-1}u_{2i})\} = \bigcup_{i=1}^{m-1} \{|f(u_{2i-1}) - f(u_{2i})|\} \\
&= \bigcup_{i=1}^{m-1} \{|(2i-1) - ((4m-1) - 2i)|\} \\
&= \bigcup_{i=1}^{m-1} \{|4i - 4m|\} = \bigcup_{i=1}^{m-1} \{4(m-i)\} \\
&= \{4(m-1), 4(m-2), ..., 4\} \\
E_2' &= \bigcup_{i=1}^{m-2} \{f(u_{2i}u_{2i+1})\} = \bigcup_{i=1}^{m-2} \{|f(u_{2i}) - f(u_{2i+1})|\} \\
&= \bigcup_{i=1}^{m-2} \{|(4m-2i-1) - (2i+1)|\} \\
&= \bigcup_{i=1}^{m-2} \{4m-4i-2\} = \{4m-6, 4m-10, ..., 6\} \\
E_3' &= \{f(u_{2m-2}u_{2m-1})\} = \{|f(u_{2m-2}) - f(u_{2m-1})|\} \\
&= \{|(2m+1) - 2|\} = \{2m-1\} \\
E_4' &= \{f(u_{2m-1}u_{2m})\} = \{|f(u_{2m-1}) - f(u_{2m})|\} \\
&= \{|2-4m|\} = \{4m-2\} \\
E_5' &= \{f(u_{2m}u_1)\} = \{|f(u_{2m}) - f(u_1)|\} = \{4m-1\}
\end{aligned}$$

In both the cases, we observe that all the vertex label sets and the edge label sets are distinct and their union is $\{1, 2, ..., 4m+2\}$ (or) $\{1, 2, ..., 4m\}$ according as $n = 2m+1$ (or) $n = 2m$. Therefore, f is a super graceful labeling and hence, any cycle $C_n (n \geq 3)$ is a super graceful graph. ■

Example 2.9. *Super graceful labelings of* C_9 *and* C_{10} *are given in Figure 7.*

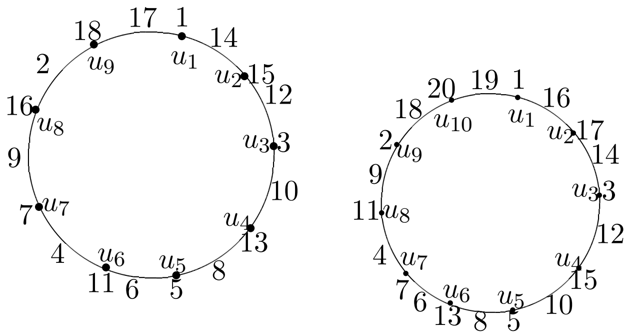

Figure 7: Super graceful labelings of C_9 and C_{10}.

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