*International Journal of Mathematics and Soft Computing*
*Vol.2, No.1 (2012), 65 - 73*

**ISSN 2249 - 3328**

# $d$-ideals and injective ideals in a distributive lattice

**M. Sambasiva Rao**
Department of Mathematics
MVGR College of Engineering, Chintalavalasa
Vizianagaram, Andhra Pradesh-535003, INDIA.
E-mail: mssraomaths35@rediffmail.com

**Abstract**

The concepts of $d$-ideals and injective ideals are introduced in a distributive lattice with respect to derivations. $d$-ideals are characterized in terms of principal ideals of a distributive lattice. Further, an equivalent condition is derived for a $d$-ideal to become an injective ideal. Also the Stone's theorem for ideals of a distributive lattice is extended to the case of injective ideals.

**Keywords:** Derivation, kernel, $d$-ideal, injective ideal, $d$-prime ideal.
**AMS Subject Classification(2010):** 06D99.

## 1 Introduction

For a ring $(R, +, \cdot)$ where $+$ and $\cdot$ denote two binary operations, we recall the derivation of $R$ as a mapping $f : R \longrightarrow R$ satisfying the following properties:

$$f(x \cdot y) = (x \cdot f(y)) + (f(x) \cdot y)$$
$$f(x + y) = f(x) + f(y)$$

H.E. Bell, L.C. Kappe [3] and K. Kaya [5] have studied derivations in rings and prime rings after Posner [6] had given the definition of the derivation in ring theory. Szasz have introduced and developed the theory of derivations in lattice structure. In a series of papers [7] and [8] he established the main properties of derivations of lattices. L. Ferrari [4] extended these concepts to lattices and he embedded any lattice having some additional properties into the lattice of its derivations.

G. Birkhoff [2], George Grätzer, G. Szász and many authors have studied about various types of ideals and congruences all intimated to some extent the behavior of ideals in a distributive lattice.

The aim of this paper is to study the structure of certain classes of ideals in a distributive lattice with respect to a derivation. On this way, the notions of $d$-ideals and injective ideals are introduced and their preliminary properties are studied in a distributive lattice. These classes of ideals are then characterized in terms of principal ideals. A necessary and sufficient condition is established for a $d$-ideal to become an injective ideal. The concept of $d$-prime ideals is introduced and the relations between the class of all injective ideals and the class of all $d$-prime ideals are obtained. Finally, the famous and crucial result of M.H. Stone is extended to the case of injective ideals and $d$-prime ideals.

## 2 Preliminaries

We give some elementary aspects and important results which are used in the sequel of this paper.

**Definition 2.1.** *[1] An algebra* $(L, \wedge, \vee)$ *of type* $(2, 2)$ *is called a distributive lattice if for all* $x, y, z \in L$, *it satisfies the following properties* $(1), (2), (3)$ *and* $(4)$ *along with* $(5)$ *or* $(5')$

(1) $x \wedge x = x, x \vee x = x$
(2) $x \wedge y = y \wedge x, x \vee y = y \vee x$
(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z), (x \vee y) \vee z = x \vee (y \vee z)$
(4) $(x \wedge y) \vee x = x, (x \vee y) \wedge x = x$
(5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(5′) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

**Remark 2.2.** *The least element of a distributive lattice is denoted by $0$. Throughout this article $L$ stands for a distributive lattice with $0$, unless otherwise mentioned.*

**Definition 2.3.** *[1] Let $(L, \wedge, \vee)$ be a lattice. A partial ordering relation $\leq$ is defined on $L$ by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$. In this case, the pair $(L, \leq)$ is called a Partially Ordered set or simply POset.*

**Definition 2.4.** *[2] A non-empty subset $A$ of $L$ is called an ideal(filter) of $L$ if $a \vee b \in A (a \wedge b \in A)$ and $a \wedge x \in A (a \vee x \in A)$ whenever $a, b \in A$ and $x \in L$.*

**Remark 2.5.** *The set $\mathcal{I}(L)$ of all ideals of a distributive lattice $L$ is a complete distributive lattice with least element $\{0\}$ and the greatest element $L$ under set inclusion in which, for any $I, J \in \mathcal{I}(L),\ I \cap J$ is the infimum of $I, J$ and the supremum is given by $I \vee J = \{\ i \vee j \mid i \in I, j \in J\ \}$. For any $a \in L,\ (a] = \{\ x \mid x \leq a\ \}$ is the principal ideal generated by $a$. The set $\mathcal{PI}(L)$ of all principal ideals of $L$ is a sublattice of the distributive lattice $\mathcal{I}(L)$.*

**Theorem 2.6.** *[2] Let $I$ be an ideal and $F$ a filter of $L$ such that $I \cap F = \emptyset$. Then there exists a prime ideal $P$ such that $I \subseteq P$ and $P \cap F = \emptyset$.*

**Definition 2.7.** *[2] Let $L$ be a lattice. The mapping $f : L \longrightarrow L$ is called a homomorphism if it satisfies the following conditions for all $x, y \in L$:*

(1) $f(x \wedge y) = f(x) \wedge f(y)$
(2) $f(x \vee y) = f(x) \vee f(y)$

A homomorphism is called injective if it is one-one.

**Definition 2.8.** *[9] A self-map $d : L \longrightarrow L$ is called a derivation of $L$ if it satisfies the following conditions:*

*(1)* $d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$
*(2)* $d(x \vee y) = d(x) \vee d(y)$

**Remark 2.9.** *In [4], Ferrari observed the condition $(1)$ is redundant and is equivalent to*

$$(1') \quad d(x \wedge y) = d(x) \wedge y = x \wedge d(y)$$

*A function satisfying the above condition $(1')$ is called a translation and these translations are extensively studied by Szasz in [7] and [8].*

**Lemma 2.10.** *[4] Let $d$ be a derivation of $L$. Then for any $x, y \in L$, we have*

*(1)* $d(0) = 0$

*(2)* $d(x) \leq x$

*(3)* $x \leq y \Rightarrow d(x) \leq d(y)$

## 3 Main Results

In this section, the concepts of $d$-ideals and injective ideals are introduced in a distributive lattice. Further, $d$-ideals are characterized in terms of principal ideals. An equivalent condition is obtained for a $d$-ideal to become an injective ideal.

**Definition 3.1.** *A self-mapping $d : L \longrightarrow L$ is called a derivation of $L$ if it satisfies the following properties:*

*(i)* $d(x \wedge y) = d(x) \wedge y$

*(ii)* $d(x \vee y) = d(x) \vee d(y)$ *for all* $x, y \in L$

The kernel of a derivation is defined as the set $Ker\ d = \{x \in L \mid d(x) = 0\}$.

**Proposition 3.2.** *For any derivation $d$ of $L$, $Ker\ d$ is an ideal of $L$.*

**Proof.** Clearly $0 \in Ker\ d$. Let $x, y \in Ker\ d$. Then $d(x \vee y) = d(x) \vee d(y) = 0$. Hence $x \vee y \in Ker\ d$. Again, let $x \in Ker\ d$ and $r \in L$. Then $d(x \wedge r) = d(x) \wedge r = 0 \wedge r = 0$. Hence $x \wedge r \in Ker\ d$. Thus $Ker\ d$ is an ideal of $L$. ∎

**Theorem 3.3.** *Let $d$ be a derivation and $I$ an ideal of $L$. Then we have*

*(i)* $d(I)$ *is an ideal of* $L$ *such that* $d(I) \subseteq I$.

*(ii)* $d^{-1}(I)$ *is an ideal of* $L$ *such that* $Ker\ d \subseteq d^{-1}(I)$.

**Proof.** (i). Clearly $0 = d(0) \in d(I)$. Let $a, b \in d(I)$. Then $a = d(x)$ and $b = d(y)$ for some $x, y \in I$. Now $a \vee b = d(x) \vee d(y) = d(x \vee y) \in d(I)$. Again, let $c \in d(I)$ and $x \in L$. Then $c = d(z)$ for some $z \in I$. Now $c \wedge x = d(z) \wedge x = d(z \wedge x) \in d(I)$. Therefore $d(I)$ is an ideal of $L$. Let $x \in d(I)$. Then $x = d(y)$ for some $y \in I$. Since $d(y) \leq y$, we get $d(y) = y \wedge d(y) \in I$. Hence $x \in I$.

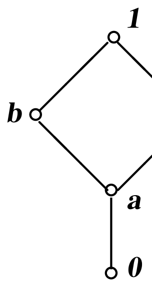
(ii). Since $d(0) = 0 \in I$, we get $0 \in d^{-1}(I)$. Let $a, b \in d^{-1}(I)$. Then we have $d(a), d(b) \in I$. Since $I$ is an ideal, we can get $d(a \vee b) = d(a) \vee d(b) \in I$. Hence $a \vee b \in d^{-1}(I)$. Again, let $x \in d^{-1}(I)$ and $r \in L$. Then we get $d(x) \in I$. Since $I$ is an ideal of $L$, we get $d(x \wedge r) = d(x) \wedge r \in I$. Thus $x \wedge r \in d^{-1}(I)$. Therefore $d^{-1}(I)$ is an ideal of $L$. Since $0 \in I$, we get $Ker\ d = d^{-1}(\{0\}) \subseteq d^{-1}(I)$. ∎

**Remark 3.4.** *One can easily observe that the condition of onto which is necessary for getting the image of an ideal under a homomorphism of distributive lattices to became again an ideal is not required in* *case of a derivation.*

**Definition 3.5.** *An ideal $I$ of $L$ is called a $d$-ideal if $I = d(I)$.*

**Remark 3.6.** *Since $d(0) = 0$, it can be easily observed that the zero ideal $\{0\}$ is a $d$-ideal of $L$. Furthermore, if $d$ is onto, then $d(L) = L$ and hence $L$ is also a $d$-ideal. However, a proper $d$-ideal is given in the following example.*

**Example 3.7.** *Consider the distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given below.*




*Define a self map $d : L \longrightarrow L$ such that $d(0) = 0, d(a) = d(c) = a$ and $d(b) = d(1) = b$. Then clearly $d$ is a derivation on $L$. Consider the ideal $I = \{0, a, b\}$. It can be verified that $d(I) = I$. Therefore, $I$ is a $d$-ideal of $L$.*

**Lemma 3.8.** *Let $d$ be a derivation of $L$ and $I, J$ any two ideals of $L$. Then we have*

(a) *$I \subseteq J$ implies that $d(I) \subseteq d(J)$.*
(b) *$d(I \cap J) = d(I) \cap d(J)$.*
(c) *$d(I \vee J) = d(I) \vee d(J)$.*

**Proof.** (a) Suppose that $I \subseteq J$. Let $x \in d(I)$. Then we get that $x = d(y)$ for some $y \in I \subseteq J$. Hence we get $x = d(y) \in d(J)$. Therefore, $d(I) \subseteq d(J)$.

(b) Clearly $d(I \cap J) \subseteq d(I) \cap d(J)$. Conversely, let $x \in d(I) \cap d(J)$. Then $x = d(a)$ for some $a \in I$ and $x = d(b)$ for some $b \in J$. Since $b \in J$ and $d(b) \leq b$, we get that $d(b) \in J$ and hence $a \wedge d(b) \in I \cap J$. Thus $x = d(a) \wedge d(b) = d(a \wedge d(b)) \in d(I \cap J)$. Therefore $d(I) \cap d(J) \subseteq d(I \cap J)$.

(c) Clearly $d(I) \vee d(J) \subseteq d(I \vee J)$. Conversely, let $x \in d(I \vee J)$. Then $x = d(z)$ for some $z \in I \vee J$. Hence $z = a \vee b$ for some $a \in I$ and $b \in J$. Thus $x = d(z) = d(a \vee b) = d(a) \vee d(b) \in d(I) \vee d(J)$. Therefore, $d(I \vee J) \subseteq d(I) \vee d(J)$. ∎

For any derivation $d$ of $L$, let us denote the class of all $d$-ideals of $L$ by $\mathcal{I}_d(L)$.

**Theorem 3.9.** *Let $d$ be a derivation of $L$. Then $\mathcal{I}_d(L)$ is a complete distributive lattice with respect to set inclusion. Moreover, $\mathcal{I}_d(L)$ has greatest element if and only if the map $d$ is onto.*

**Proof.** Define an order $\leq$ on $\mathcal{I}_d(L)$ by $I \leq J$ if and only if $I \subseteq J$ for any two $I, J \in \mathcal{I}_d(L)$. Then clearly $(\mathcal{I}_d(L), \leq)$ is a partially ordered set. Since $\{0\}$ is a $d$-ideal, it can be easily obtained that

$(\mathcal{I}_d(L), \leq)$ is a complete lattice. Again by the above lemma, it yields that $\langle \mathcal{I}_d(L), \cap, \vee \rangle$ is a sublattice of $\mathcal{I}(L)$ of all ideals of $L$. Hence $\langle \mathcal{I}_d(L), \cap, \vee, \{0\} \rangle$ is a complete distributive lattice. The remaining part is clear by the observation that $d$ is onto if and only if $d(L) = L$. ■

**Theorem 3.10.** *Let $d$ be a derivation of $L$. Then for any ideal $I$ of $L$, the following conditions are equivalent.*

*(1) $I$ is a $d$-ideal.*

*(2) $I = \bigcup\limits_{x \in I} (d(x)]$.*

*(3) For any $x \in I$, there exists $y \in I$ such that $x = d(y)$.*

**Proof.** $(1) \Rightarrow (2)$: Since $d(x) \leq x$, we have always $\bigcup\limits_{x \in I} (d(x)] \subseteq \bigcup\limits_{x \in I} (x] = I$. Conversely, let $x \in I$. Then there exists $a \in I$ such that $x = d(a)$. Hence $x \in (d(a)] \subseteq \bigcup\limits_{t \in I} (d(t)]$. Thus we get $I \subseteq \bigcup\limits_{t \in I} (d(t)]$. Therefore $I = \bigcup\limits_{t \in I} (d(t)]$.

$(2) \Rightarrow (3)$: Assume the condition (2). Let $x \in I$. Then $x \in (d(a)]$ for some $a \in I$. Hence $x = d(a) \wedge x = d(a \wedge x)$. Therefore $x = d(a \wedge x)$ and $a \wedge x \in I$.

$(3) \Rightarrow (1)$: Assume the condition (3). We have always $d(I) \subseteq I$. Now let $x \in I$. Then there exists $y \in I$ such that $x = d(y) \in d(I)$. Therefore $I = d(I)$. ■

**Definition 3.11.** *Let $d$ be a derivation of $L$. An ideal $I$ of $L$ is called an injective ideal with respect to $d$ if for $x, y \in L$, $d(x) = d(y)$ and $x \in I$ implies that $y \in I$.*

Evidently $Ker\ d$ is an injective ideal of $L$. Though the zero ideal $\{0\}$ is a $d$-ideal, there is no guarantee that it is an injective ideal. However, a set of equivalent conditions are established for $\{0\}$ to become an injective ideal.

**Proposition 3.12.** *Let $d$ be a derivation of $L$. Then the following conditions are equivalent.*

*(a) $\{0\}$ is injective with respect to $d$.*

*(b) $Ker\ d = \{0\}$.*

*(c) $d(x) = 0$ implies that $x = 0$ for all $x \in L$.*

**Proof.** $(a) \Rightarrow (b)$: Assume that $\{0\}$ is injective with respect to $d$. Let $x \in Ker\ d$. Then $d(x) = 0 = d(0)$. Since $\{0\}$ is injective, we can get that $x \in \{0\}$. Therefore $Ker\ d = \{0\}$.

$(b) \Rightarrow (c)$: The proof is trivial.

$(c) \Rightarrow (a)$: Assume the condition $(c)$. Let $d(x) = d(y)$ and $x \in \{0\}$. Hence $d(y) = d(x) = d(0) = 0$. Therefore condition (c) yields that $y = 0 \in \{0\}$. ■

**Theorem 3.13.** *Let $d$ be a derivation of $L$. Then the following conditions are equivalent:*

*(1) $d$ is injective.*

*(2) Every ideal is injective with respect to $d$.*

*(3) Every prime ideal is injective with respect to $d$.*

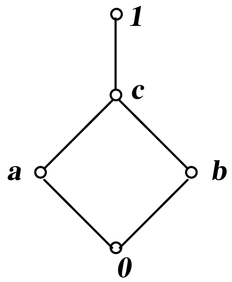
**Proof.** $(1) \Rightarrow (2)$: It is clear by the definition of an injective ideal.

$(2) \Rightarrow (3)$: Obvious.

$(3) \Rightarrow (1)$: Assume that every prime ideal of $L$ is an injective ideal. Let $x, y \in L$ be such that $d(x) = d(y)$. Suppose that $x \neq y$. Without loss of generality, we can assume that $(x] \cap [y) = \emptyset$. Then there exists a prime ideal $P$ such that $x \in P$ and $y \notin P$, which is a contradiction to $P$ is an injective ideal. ∎

The independency between the class of all $d$-ideals and that of injective ideals are demonstrated in the following examples.

**Example 3.14.** *Consider the distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given below.*




*Define a self-map $d : L \longrightarrow L$ as follows:*

$$d(x) = \begin{cases} a & \text{if } x = a, c, 1 \\ 0 & \text{Otherwise} \end{cases}$$

*Then it can be easily verified that $d$ is a derivation of $L$. $I = \{0, b\}$ is an injective ideal of $L$. Now $d(I) = \{0\}$ and hence $I \neq d(I)$. Therefore, $I$ is an injective ideal with respect to $d$ but not a $d$-ideal.*

**Example 3.15.** *Consider the distributive lattice $L = \{0, a, b, c, 1\}$ given in example 2.7. Define a self map $d : L \longrightarrow L$ as follows:*

$$d(x) = \begin{cases} 0 & \text{if } x = 0 \\ a & \text{if } x = a, c \\ b & \text{if } x = b, 1 \end{cases}$$

*It can be easily verified that $d$ is a derivation of $L$. Now consider the ideal $I = \{0, a, b\}$. Clearly $d(I) = I$ and hence $I$ is a $d$-ideal. But $d(a) = d(c), a \in I$ and $c \notin I$. Therefore, $I$ is a $d$-ideal but not injective with respect to $d$.*

**Theorem 3.16.** *Let $d$ be a derivation of $L$. Then a $d$-ideal $I$ of $L$ is injective with respect to $d$ if and only if for any $x \in L$, $d(x) \in I$ implies $x \in I$.*

**Proof.** Let $I$ be a $d$-ideal of $L$. Assume that $I$ is an injective ideal with respect to $d$. Let $x \in L$. Suppose $d(x) \in I = d(I)$. Then $d(x) = d(a)$ for some $a \in I$. Since $I$ is injective and $a \in I$, we get that $x \in I$. Conversely, let $x, y \in L, d(x) = d(y)$ and $x \in I$. Since $x \in I = d(I)$, we get $x = d(a)$ for some $a \in I$. Hence, $d(y) = d(x) = d(d(a))$ and $d(a) \leq a \in I$ which implies that $y \in I$. Therefore, $I$ is an injective ideal of $L$ with respect to $d$. ∎

**Definition 3.17.** *Let $d$ be a derivation of $L$. Then for any ideal $I$ of $L$, define an extension of $I$ as $I' = \{x \in L \mid d(x) \in (d(a)] \text{ for some } a \in I\}$.*

The following lemma is a routine verification.

**Lemma 3.18.** *Let $d$ be a derivation of $L$. Then for any two ideals $I, J$ of $L$, we have the following:*

(1) *$I'$ is an ideal of $L$.*
(2) *$I \subseteq I'$.*
(3) *$I \subseteq J$ implies $I' \subseteq J'$.*
(4) *$I' \cap J' = (I \cap J)'$.*

**Proposition 3.19.** *Let $d$ be a derivation of $L$. Then for any ideal $I$ of $L$, $I'$ is the smallest injective ideal with respect to $d$ such that $I \subseteq I'$.*

**Proof.** By the above lemma, $I'$ is an ideal containing $I$. We now show that $I'$ is an injective ideal with respect to $d$. Let $x, y \in L, d(x) = d(y)$ and $x \in I'$. Then $d(y) = d(x) \in (d(a)]$ for some $a \in I$. Thus $y \in I'$. Let $J$ be an injective ideal with respect to $d$ such that $I \subseteq J$. Let $t \in I'$. Then $d(t) \in (d(a)]$ for some $a \in I \subseteq J$. Hence, $d(t) = d(a) \wedge d(t) = d(a \wedge d(t))$ and $a \wedge d(t) \in J$. Since $J$ is injective with respect to $d$, we get that $t \in J$. Thus $I' \subseteq J$ and hence $I'$ is the smallest injective ideal with respect to $d$ such that $I \subseteq I'$. ∎

**Corollary 3.20.** *If $I$ is an injective ideal, then $I = I'$.*

**Theorem 3.21.** *The set $\mathcal{I}^d(L)$ of all injective ideals of $L$, with respect to a given derivation of $d$ of $L$, forms a distributive lattice on their own.*

**Proof.** For $I, J \in \mathcal{I}^d(L)$, define the operations $\wedge$ and $\sqcup$ such that $I \wedge J = I \cap J$ and $I \sqcup J = (I \vee J)'$. Clearly $(\mathcal{I}^d(L), \wedge, \sqcup)$ is a lattice. Now for any $I, J, K \in \mathcal{I}^d(L)$ we have $I \sqcup (J \cap K) = \{I \vee (J \cap K)\}' = \{(I \vee J) \cap (I \vee K)\}' = (I \vee J)' \cap (I \vee K)' = (I \sqcup J) \cap (I \sqcup K)$. Therefore, $(\mathcal{I}^d(L), \wedge, \sqcup)$ is a distributive lattice. ∎

**Definition 3.22.** *Let $d$ be a derivation of $L$. A proper ideal $P$ of $L$ is called a $d$-prime ideal if for any $x, y \in L$, $x \wedge y \in Ker\ d$ implies either $x \in P$ or $y \in P$.*

By a maximal injective ideal, we mean an injective ideal which is maximal in the class of all proper injective ideals with respect to a given derivation. Then the following series of propositions establish some useful relations between the class of all injective ideals with respect to a derivation $d$ and the class of all $d$-prime ideals. For this, the following lemma is required.

**Lemma 3.23.** *Let $d$ be a derivation of $L$. Then for any injective ideal $I$ of $L$ with respect to $d$ , $Ker\ d \subseteq I$. In other words, $Ker\ d$ is the smallest injective ideal of $L$.*

**Proof.** Let $I$ be an injective ideal with respect to $d$ of $L$. Suppose $x \in Ker\ d$. Then $d(x) = 0 = d(0)$. Since $I$ is injective with respect to $d$ and $0 \in I$, it yields that $x \in I$. Since $Ker\ d$ is also an injective ideal with respect to $d$ of $L$, it is the smallest injective ideal in $L$ with respect to the derivation $d$. ∎

**Proposition 3.24.** *Let $d$ be a derivation of $L$. Then every maximal injective ideal with respect to the derivation $d$ of $L$ is a $d$-prime ideal.*

**Proof.** Let $M$ be a maximal injective ideal of $L$. Choose $x, y \in L$ such that $x \notin M$ and $y \notin M$. Then $M \subset M \vee (x] \subseteq \{M \vee (x]\}'$ and $M \subset M \vee (y] \subseteq \{M \vee (y]\}'$. Since $M$ is maximal, we can get $\{M \vee (x]\}' = L$ and $\{M \vee (y]\}' = L$. Hence

$$\begin{aligned}\{M \vee (x \wedge y]\}' &= \{(M \vee (x]) \bigcap (M \vee (y])\}' \\ &= \{M \vee (x]\}' \bigcap \{M \vee (y]\}' \\ &= L\end{aligned}$$

If $x \wedge y \in Ker\ d$, then $L = \{M \vee (x \wedge y]\}' \subseteq \{M \vee Ker\ d\}' = M' = M$, which is a contradiction to the fact that $M$ is proper. Hence $M$ is a $d$-prime ideal. ∎

**Proposition 3.25.** *Let $d$ be a derivation of $L$. Then every $d$-prime ideal $P$ is an injective ideal with respect to $d$ if for each $a \in P$ there exists $b \notin P$ such that $a \wedge b \in Ker\ d$.*

**Proof.** Let $P$ be a $d$-prime ideal which satisfies the given property. Let $x, y \in L$ be such that $d(x) = d(y)$. Suppose $x \in P$. Then there exists $x' \notin P$ such that $x \wedge x' \in Ker\ d$. Hence $d(x) \wedge x' = d(x \wedge x') = 0$, which yields that $d(y \wedge x') = d(y) \wedge x' = 0$. Therefore $y \wedge x' \in Kerd$. Since $P$ is a $d$-prime ideal and $x' \notin P$, we get that $y \in P$. Hence, $P$ is injective with respect to $d$. ∎

We generalise the celebrated result of M.H. Stone [1936] which is meant for ideals, filters and prime ideals of a distributive lattice to the case of injective ideals, filters and $d$-prime ideals.

**Theorem 3.26.** *Let $d$ be a derivation of $L$. Suppose $I$ is an injective ideal with respect to $d$ and $F$ a filter of $L$ such that $I \cap F = \emptyset$. Then there exists a $d$-prime ideal $P$ such that $I \subseteq P$ and $P \cap F = \emptyset$.*

**Proof.** Let $I$ be an injective ideal with respect to $d$ and $F$ a filter of $L$ such that $I \cap F = \emptyset$. Consider $\mathfrak{F} = \{J \in I^d(L) \mid I \subseteq J \text{ and } J \cap F = \emptyset\}$. Clearly $I \in \mathfrak{F}$. Let $\{J_\alpha\}$ be a chain of elements of $\mathfrak{F}$. Then clearly $\cup J_\alpha$ is an upper bound for $\{J_\alpha\}$ in $\mathfrak{F}$. Hence by Zorn's lemma, $\mathfrak{F}$ has a maximal element, say $M$. We now prove that $M$ is $d$-prime. Let $x, y \in L$ such that $x \notin M$ and $y \notin M$. Then $M \subset M \vee (x] \subseteq \{M \vee (x]\}'$ and $M \subset M \vee (y] \subseteq \{M \vee (y]\}'$. By the maximality of $M$, we get that

$\{M \vee (x]\}' \cap F \neq \emptyset$ and $\{M \vee (y]\}' \cap F \neq \emptyset$. Choose $a \in \{M \vee (x]\}' \cap F$ and $b \in \{M \vee (y]\}' \cap F$. Then

$$\begin{aligned} a \wedge b &\in \{M \vee (x]\}' \bigcap \{M \vee (y]\}' \\ &= \{(M \vee (x]) \bigcap (M \vee (y])\}' \\ &= \{M \vee (x \wedge y]\}' \end{aligned}$$

Suppose $x \wedge y \in Ker\ d$. Then it reflects that $a \wedge b \in \{M \vee (x \wedge y]\}' = \{M \vee Ker\ d\}' = M' = M$(Since $M$ is injective). Thus, $a \wedge b \in M \cap F$, which is a contradiction to $M \cap F = \emptyset$. Therefore, $M$ is a $d$-prime ideal. ∎

## References


[1] R. Balbes and P. Dwinger, *Distributive Lattices*, University of Missouri Press, Columbia, Mo., (1974).

[2] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloq. XXV, Providence, U.S.A., (1967).

[3] H.E. Bell and L.C. Kappe, *Rings in which derivations satisfy certain algebraic conditions*, Acta Math. Hungar., 53(3-4)(1989), 339-346.

[4] L. Ferrari, *On derivations of lattices*, Pure Mathematics and Applications, 12(2001), No.45, 365-382.

[5] K. Kaya, *Prime rings with a derivations*, Bull. Mater. Sci. Eng., 16(1987), 63-71.

[6] E. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc., 8(1957), 1093-1100.

[7] G. Szász, *Translation der Verbände*, Acta Fac. Rer. Nat. Univ. Comenianae, 5(1961), 53-57.

[8] G. Szász, *Derivations of lattices*, Acta Sci. Math.(Szeged), 37(1975), 149-154.

[9] X.L. Xin, T.Y. Li and J.H. Lu, *On derivations of lattices*, Inform. Sci., 178(2008), No. 2, 307-316.