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On super edge-magic total labeling of certain classes of graphs

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Abstract

A (p, q)- simple graph is edge-magic if there exists a bijective function $\lambda : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p + q\}$ such that $\lambda(u) + \lambda(uv) + \lambda(v) = k$, for all edge $uv \in E(G)$, where k is called the magic constant or sometimes the valence of λ. An edge-magic total labeling λ is called super edge-magic total if $\lambda(V(G)) = \{1, 2, \ldots, p\}$. In this paper, we construct new classes of trees using w- trees and generalized combs and prove that they admit super edge magic total labeling. We also prove that the extended umbrella graphs admit super edge-magic total labeling.

Keywords: Super edge magic total labeling, umbrella graphs, comb graphs, w-trees.
AMS Subject Classification(2010): 05C78.

1 Introduction and Preliminary Results

The graph labeling has caught the attention of several authors and new results on labeling appear every year. This popularity is not only due to the mathematical challenges of graph labeling, but also for the wide range of its application, for instance X-ray, crystallography, coding theory, radar, astronomy, circuit 1 design, network design and communication design. Let $G(V, E)$ be a finite, simple and undirected graph with $|V(G)| = p$ and $|E(G)| = q$. Kotzig and Rosa [14] defined a magic labeling λ on a graph G to be a bijection that assigns the distinct integers from 1 to $p + q$ to all the vertices and edges of the graph such that the sums of the labels for an edge and its two endpoints is constant for each edge. Ringel and Lladó [18] redefined this type of labeling as *edge-magic*. Recently, Enomoto et al. [3] used the name *super edge-magic* for the magic labelings defined by Kotzig and Rosa, with an additional property that the vertices receive the smallest labels. That is, $\lambda(V(G)) = \{1, 2, 3, \ldots, p\}$.

If the domain of a labeling λ is the set of all vertices and edges of the graph G, then such labeling is called *total labeling*. In this paper, we study the labelings which have another property that the weight $\omega(xy)\ \forall\ xy \in E(G)$, calculated as; $\omega(xy) = \lambda(x) + \lambda(y) + \lambda(xy)$, is equal to a fixed constant k, called the *magic constant* or sometimes the $valence$ of λ. A graph is called *super edge-magic total* (SEMT) if it admits a *super edge-magic total labeling*. Some labelings have only the vertex-set (edge-set) as their domain, such labelings are called *vertex-labelings* (*edge-labelings*). Other domains for the labeling λ are also possible.

A number of classification problems on SEMT labeling of connected graphs have been investigated. Figueroa-Centeno et al. [5] proved the following:

- If G is a bipartite or tripartite (super) edge-magic graph, then nG is (super) edge-magic when n is odd.
- If m is a multiple of $n+1$, then $St(m) \cup St(n)$ is super edge-magic.
- $St(2) \cup St(n)$ is super edge-magic if and only if n is a multiple of 3.
- $P_m \cup St(n)$ is super edge-magic when $m \geq 4$.
- $2P_n$ is super edge-magic if and only if n is not 2 or 3.
- $St(m) \cup 2nSt(2)$ is super edge-magic for all m and n.

Lee and Kong [16] used the notation $St(a_1, a_2, \ldots, a_n)$ to denote the disjoint union of n stars $St(a_1)$, $St(a_2)$,..., $St(a_n)$ of order $a_1+1, a_2+1, \ldots, a_n+1$, respectively. They proved the following to be super edge-magic:

- $St(m, n)$ where $n \equiv 0 \mod (m+1)$.
- $St(1,1,n)$, $St(1,2,n)$, $St(1,n,n)$, $St(2,2,n)$, $St(2,3,n)$, $St(1,1,2,n)$ for $n \geq 2$.

It is known that if a (p,q)-graph G is super edge-magic, then $q \leq 2p-3$ [3]. This bound can be improved for bipartite graphs of order $p \geq 4$, to be $q \leq 2p-5$ [17]. For more results concerning edge-magic total labelings, one can refer [1, 6–8].

All the theorems in this paper are proved using the Lemma proposed by Figueroa et al. [4], stated below.

Lemma 1.1. A (p,q) graph G is super edge-magic total if and only if there exists a bijection $\lambda : V(G) \to \{1, 2, \cdots, p\}$ such that the set of edge weights $S = \{\lambda(x) + \lambda(y) | xy \in E(G)\}$ consists of q consecutive integers. In such a case, λ extends to a super edge-magic total labeling of G with magic constant $k = p + q + s$, where $s = min(S)$ and

$$S = \{k-(p+1), k-(p+2), k-(p+3), \ldots, k-(p+q)\}.$$

2 Main Results

In this section, we present the main results about the new graph classes constructed using the old ones and study their super edge-magic total labeling schemes.

2.1 SEMT labeling of generalized w-tree

Javaid et al. [11] studied the super edge-magic total labeling of *w-graphs* ($W(n)$) and *w-trees* ($WT(n,k)$). They have defined a w-graph $W(n)$ to be a graph obtained by identifying a pendant vertex from two isomorphic copies of a star $St(n+3)$, and a w-tree $WT(n,k)$ to be a graph obtained by joining one vertex each from k isomorphic copies of $W(n)$ to a new vertex a.

Let $W(n_1, n_2)$ be a graph obtained by identifying a pendant vertex of the star $St(n_1+2)$ with a pendant vertex of $St(n_2+2)$. We define a *generalized w-tree* in the following way.

Definition 2.1. A generalized w-tree $WT(n_1, n_2, \ldots, n_{2k}; k)$ is a graph derived from $W(n_i, n_{i+1})$ for $1 \leq i \leq 2k$, $i \equiv 1\ (mod\ 2)$, by joining a pendant vertex from each $W(n_i, n_{i+1})$ to a new vertex a.

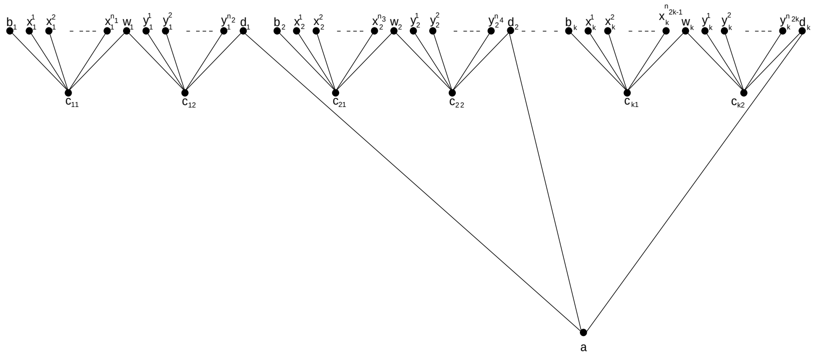

Figure 1: Generalized w-tree.

Theorem 2.2. The graph $G \cong WT(n_1, n_2, ..., n_{2k}; k)$ for $n_j \geq 2$ whenever $j \equiv 2 \pmod 4$ and $k \in \mathbb{N}$ admits super edge magic total labeling.

Proof: The vertex and the edge sets of the graph $G \cong WT(n_1, n_2, ..., n_{2k}; k)$ are given by

$$V(G) = \{a\} \cup \{b_i, w_i, d_i, c_{i1}, c_{i2};\ 1 \leq i \leq k\} \cup \{x_i^l,\ 1 \leq i \leq k,\ 1 \leq l \leq n_{2i-1}\}$$
$$\cup\{y_i^l,\ 1 \leq i \leq k,\ 1 \leq l \leq n_{2i}\}$$
$$E(G) = \{b_i\, c_{i1}, d_i\, c_{i2}, w_i\, c_{i1}, w_i\, c_{i2}, a\, d_i\,;\ 1 \leq i \leq k\} \cup \{c_{i1}\, x_i^l\,;\ 1 \leq i \leq k,\ 1 \leq l \leq n_{2i-1}\}$$
$$\cup\{c_{i2}\, y_i^l\,;\ 1 \leq i \leq k,\ 1 \leq l \leq n_{2i}\}.$$

The order of the graph G is $\nu = \sum_{i=1}^{2k} n_i + 5k + 1$ and size $\epsilon = \sum_{i=1}^{2k} n_i + 5k$.

Let $s = \lfloor \frac{k}{2} \rfloor$. We define the labeling $\lambda : V(G) \to \{1, 2, \ldots, \sum_{i=1}^{2k} n_i + 5k + 1\}$ as follows:

$$\lambda(a) = \sum_{i=1}^{2k} n_i + 5k + 1 - 2\lceil \tfrac{k}{2} \rceil.$$

$$\lambda(c_{i1}) = \begin{cases} \nu - 2k + 2i - 2, & 1 \leq i \leq s, \\ \nu - 2k + 2i, & s + 1 \leq i \leq k. \end{cases}$$

$$\lambda(c_{i2}) = \sum_{i=1}^{2k} n_i + 3k + 2i\,;\ 1 \leq i \leq k.$$

$$\lambda(b_i) = \begin{cases} 1, & \text{i=1, s=1}, \\ \sum_{t=1}^{2i-2} n_t + 3i - 2, & 2 \leq i \leq s, \\ \sum_{t=1}^{2i} n_t + 3i, & s + 1 \leq i \leq k. \end{cases}$$

$$\lambda(w_i) = \begin{cases} \sum_{t=1}^{2i-1} n_t + 3i - 1, & 1 \le i \le s, \\ \sum_{t=1}^{2i} n_t - n_{2i-1} + 3i - 1, & s+1 \le i \le k. \end{cases}$$

$$\lambda(d_i) = \begin{cases} \lambda(w_i) + 2i + n_{2i} - 2s + 1, & 1 \le i \le s, \\ \lambda(w_i) + 2i + n_{2i} - 2s - 3, & s+1 \le i \le k. \end{cases}$$

$$\lambda(x_i^l) = \begin{cases} \{\lambda(b_i)+1, \lambda(b_i)+2, \ldots, \lambda(b_i)+n_{2i-1}\}, & 1 \le i \le s, \\ \{\lambda(b_i)-1, \lambda(b_i)-2, \ldots, \lambda(b_i)-n_{2i-1}\}, & s+1 \le i \le k. \end{cases}$$

$$\lambda(y_i^l) = \begin{cases} \{\lambda(w_i)+1, \lambda(w_i)+2, \ldots, \lambda(w_i)+n_{2i}, \lambda(w_i)+n_{2i}+1\} \setminus \lambda(d_i), & 1 \le i \le s, \\ \{\lambda(w_i)-1, \lambda(w_i)-2, \ldots, \lambda(w_i)-n_{2i}, \lambda(w_i)-n_{2i}-1\} \setminus \lambda(d_i), & s+1 \le i \le k. \end{cases}$$

The edge weights of G form a sequence of ϵ consecutive integers, which are $\left\{\sum_{i=1}^{2k} n_i + 3k + 2, \sum_{i=1}^{2k} n_i + 3k + 3, \ldots, 2\sum_{i=1}^{2k} n_i + 8k + 1\right\}$. So by Lemma 1.1, the graph $G \cong WT(n_1, n_2, \ldots, n_{2k}; k)$ is *super edge magic total* with magic constant $3\sum_{i=1}^{2k} n_i + 13k + 3$. ■

2.2 SEMT labeling of umbrella and extended umbrella graphs

Sin-Min Lee and Nien-Tsu Lee [15] defined an *umbrella graph* $U(m,n)$ to be a graph obtained by joining a path P_n with the central vertex of a fan f_m. In Theorem 2.3, we prove that $U(m,n)$ admits SEMT labeling for the particular values of m and n.

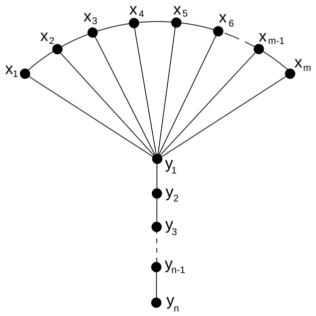

Figure 2: $U(m,n)$.

Theorem 2.3. The graph $G \cong U(m,n)$ admits super edge magic total labeling, for any $m \in \mathbb{Z}^+$ and
$$n = \begin{cases} m, m-1, & m \equiv 1 (mod\, 2); \\ m-1, m-2, & m \equiv 0 (mod\, 2). \end{cases}$$

Proof: The order and size of the graph $G \cong U(m,n)$ are $m+n$ and $2m+n-2$, respectively. The vertex and edge sets of G are given as follows.

$$V(G) = \{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n\},$$
$$E(G) = \{x_i x_{i+1} : 1 \le i \le m-1\} \cup \{y_i y_{i+1} : 1 \le i \le n-1\} \cup \{x_i y_1 : 1 \le i \le m\}.$$

To prove the existence of a super edge magic total labeling, we consider the following two cases.
Case 1: When m is odd.
Define $f : V(G) \rightarrow \{1, 2, \ldots, m+n\}$ as follows:

$$\begin{aligned} f(x_{2i-1}) &= i, \qquad i = 1,2,3,\ldots,\lfloor\frac{m}{2}\rfloor+1. \\ f(x_{2i}) &= \lfloor\frac{m}{2}\rfloor+i+1, \qquad i = 1,2,3,\ldots,\lfloor\frac{m}{2}\rfloor. \\ f(y_{2j-1}) &= m+\lfloor\frac{n}{2}\rfloor+j, \qquad j = 1,2,3,\ldots,\lfloor\frac{n}{2}\rfloor+1. \\ f(y_{2j}) &= m+j, \qquad j = 1,2,3,\ldots,\lfloor\frac{n}{2}\rfloor. \end{aligned}$$

All the edge weights of $U(m, n)$ under this labeling scheme form a sequence of consecutive integers:

$$\Big\{\lceil\frac{m}{2}\rceil+2, \lceil\frac{m}{2}\rceil+3, \ldots, \lceil\frac{m}{2}\rceil+2m+n-1\Big\}.$$

Hence, this labeling can be extended to the *SEMT labeling*, by using Lemma 1.1. The magic constant under this labeling is $\lceil\frac{m}{2}\rceil + 3m + 2n$.
Case 2: When m is even.
Define $f : V(G) \rightarrow \{1, 2, \ldots, m+n\}$ as follows:

$$\begin{aligned} f(x_{2i-1}) &= i, \qquad i = 1,2,3,\ldots,\lfloor\frac{m}{2}\rfloor. \\ f(x_{2i}) &= \lfloor\frac{m}{2}\rfloor+i, \qquad i = 1,2,3,\ldots,\lfloor\frac{m}{2}\rfloor. \\ f(y_{2j-1}) &= m+\lfloor\frac{n}{2}\rfloor+j, \qquad j = 1,2,3,\ldots,\lfloor\frac{n}{2}\rfloor. \\ f(y_{2j}) &= m+j, \qquad j = 1,2,3,\ldots,\lfloor\frac{n}{2}\rfloor. \end{aligned}$$

In this case, the edge weights of the graph $U(m, n)$ form a sequence of consecutive integers, which are:

$$\Big\{\frac{m}{2}+2, \frac{m}{2}+3, \ldots, \frac{m}{2}+2m+n-1\Big\}.$$

Hence by using Lemma 1.1, the labeling f of this graph can be converted into the *SEMT labeling*. The magic constant under this labeling is $\frac{m}{2} + 3m + 2n$. ∎

Definition 2.4. An *extended umbrella graph* $U(m, n, k)$ is a graph constructed by identifying the pendant vertex of umbrella $U(m, n)$ with the center of the star $St(k)$.

Theorem 2.5. The graph $G \cong U(m, n, k)$ admits *super edge magic total labeling*.

Proof: The order and size of the graph $G \cong U(m, n, k)$ are $m+n+k$ and $2m+n+k-2$, respectively. The vertex and edge sets of G are defined as follows.

$$V(G) = \{x_i : 1 \leq i \leq m\} \cup \{y_i : 1 \leq i \leq n\} \cup \{z_i : 1 \leq i \leq k\}.$$

$$\begin{aligned} E(G) &= \{x_ix_{i+1} : 1 \le i \le m-1\} \cup \{y_iy_{i+1} : 1 \le i \le n-1\} \\ &\cup \{x_iy_1 : 1 \le i \le m\} \cup \{z_iy_n : 1 \le i \le k, \text{ when m} \equiv 1 (\text{mod } 2) \\ &\text{and n} = \text{m} - 1, \text{ or, when m} \equiv 0 (\text{mod } 2) \text{ and n} = \text{m} - 2\} \\ &\cup \{z_iy_{n-1} : 1 \le i \le k, \text{ when m} \equiv 1 (\text{mod } 2) \text{and n} = \text{m}, \text{ or,} \\ &\text{when m} \equiv 0 (\text{mod } 2) \text{ and n} = \text{m} - 1\}. \end{aligned}$$

Define a bijection $f' : V(G) \to \{1, 2, \ldots, m+n+k\}$ such that the vertices x_i and y_i of G are labeled under f' with the same labels as the labels of the vertices x_i and y_i under *super edge magic total labeling* (f) of $U(m, n)$, in Theorem 2.3.
For the vertices z_i of the star, define $f'(z_i) = m + n + i, \quad 1 \le i \le k.$

When z_i is adjacent to y_n, all the edge weights in this labeling function form a sequence of consecutive integers: $\left\{ \lceil \frac{m}{2} \rceil + 2, \lceil \frac{m}{2} \rceil + 3, \ldots, \lceil \frac{m}{2} \rceil + 2m + n + k - 1 \right\}$.

Hence, by Lemma 1.1, the graph $U(m, n, k)$ is SEMT. The magic constant under this labeling is $\lceil \frac{m}{2} \rceil + 3m + 2n + 2k$.
When z_i is adjacent to y_{n-1}, the edge weights under the labeling function f' forms a sequence of consecutive integers: $\left\{ \frac{m}{2} + 2, \frac{m}{2} + 3, \ldots, \frac{m}{2} + 2m + n + k - 1 \right\}$.

So, again by Lemma 1.1, the graph $U(m, n, k)$ is SEMT. The magic constant under this labeling is $\frac{m}{2} + 3m + 2n + 2k$. ∎

Example 2.6. The super edge magic total labeling of extended umbrella graph $U(10, 9, 8)$ is presented in Figure 3. The magic constant of $U(10, 9, 8)$ under this labeling is 69.

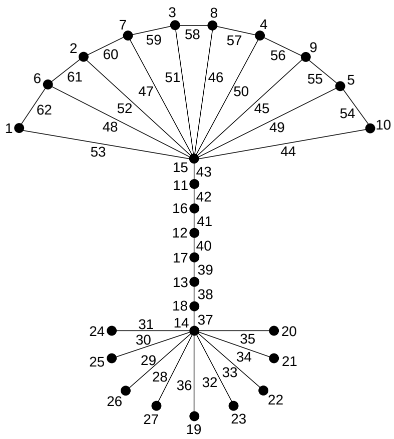

Figure 3: $U(10, 9, 8)$ with magic constant 69.

2.3 SEMT labeling of two generalized combs

A *generalized comb* [12], denoted as $Cb_n(l_1, l_2, \ldots, l_m)$, is a graph constructed from a path P_{m+1} : $x_{1,j}$ with $0 \le j \le m$ $(m \ge 2)$, and m paths P_{l_i} : $x_{t,i}$ with $1 \le i \le m$, $1 \le t \le l_i$ $(l_i \ge 2)$, by identifying the vertex $x_{1,j}$ of P_{m+1} with the vertex $x_{1,i}$ of P_{l_i}, for $1 \leq i, j \leq m$, respectively. In Figure 4, we show the special case of generalized comb when $l_i = 4$ for $1 \leq i \leq 4$.

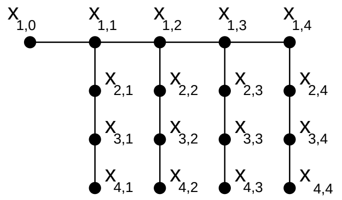

Figure 4: $Cb_n(4, 4, 4, 4)$.

Theorem 2.7. The graph $G \cong 2Cb_n(l, l, \ldots, l) + \{e\}$ admits super edge magic total labeling, for $l \geq 1$, $n \geq 2$ is even and $e = x^1_{2,n}\, x^2_{1,0}$.

Proof: The two isomorphic copies of generalized comb $Cb_n(l, l, ..., l)$ are super edge magic total if we add an edge to it. The order and size of this graph is $2(ln + 1)$ and $2ln + 1$ respectively. The vertex and edge set of this graph are $V(G) = \{x^k_{i,j} \,:\, 1 \leq i \leq l, 1 \leq j \leq n, 1 \leq k \leq 2\} \cup \{x^k_{1,0}\}$

$$\begin{aligned} E(G) &= \{x^k_{i,j}\, x^k_{i+1,j} \,:\, 1 \leq i \leq l-1, 1 \leq j \leq n, 1 \leq k \leq 2\} \cup \\ &\quad \{x^k_{i,j}\, x^k_{i,j+1} \,:\, 0 \leq j \leq n-1, 1 \leq k \leq 2\} \cup \{e\}. \end{aligned}$$

where e is the edge as defined in the statement of this theorem.
Define a labeling f such that for $1 \leq k \leq 2$,
$f(x^k_{1,0}) \;=\; (k+1)\lceil \frac{nl}{2} \rceil - 1,$

$$f(x^k_{i,j}) = \begin{cases} \lceil \frac{nl}{2} \rceil + \frac{jl+2-i}{2}, & 2 \leq i \leq l\,(even), 2 \leq j \leq n\,(even), \\ \lceil \frac{nl}{2} \rceil + \frac{(i+1)+l(j-1)}{2}, & 1 \leq i \leq l\,(odd), 1 \leq j \leq n\,(odd), \\ (k+1)\lceil \frac{nl}{2} \rceil + \frac{(i+1)+l(j-1)}{2}, & 2 \leq i \leq l\,(even), 1 \leq j \leq n\,(odd), \\ (k+1)\lceil \frac{nl}{2} \rceil + \frac{(3-i)+lj}{2}, & 1 \leq i \leq l\,(odd), 2 \leq j \leq n\,(even). \end{cases}$$

Hence the theorem is proved. ■

The labeling scheme of the above theorem is shown in Figure 5.

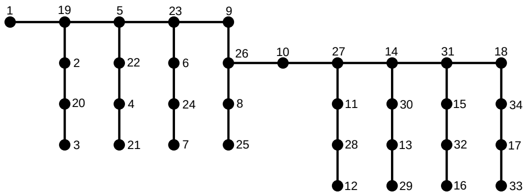

Figure 5: Generalized Comb.

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