International Journal of Mathematics and Soft Computing
Vol.4, No.1. (2014), 129 - 137.

ISSN Print : 2249 – 3328
ISSN Online: 2319 – 5215

Square graceful graphs

T. Tharmaraj
Department of Mathematics,
Udaya School of Engineering, Vellamodi,
Tamil Nadu, INDIA.
E-mail:trajtr@gmail.com.

P. B. Sarasija
Department of Mathematics,
Noorul Islam University, Kumaracoil,
Tamil Nadu, INDIA.
E-mail:sijavk@gmail.com.

Abstract

A (p,q) graph $G(V,E)$ is said to be a square graceful graph if there exists an injection $f:V(G)\to\{0,1,2,3,...,\ q^2\}$ such that the induced mapping $f_p:\ E(G)\ \to\ \{1,4,\ 9,...,q^2\}$ defined by $f_p(uv)=|f(u)-f(v)|$ is a bijection. The function f is called a square labeling of G. In this paper, we prove that the star $K_{1,n}$, bistar $B_{m,n}$, the graph obtained by the subdivision of the edges of the star $K_{1,n}$, the graph obtained by the subdivision of the central edge of the bistar $B_{m,n}$, the generalised crown $C_3\Theta K_{1,n}$, graph $P_m\Theta nK_1$ $(n\geq 2)$, the comb $P_n\Theta K_1$, graph (P_m,S_n), $(3,t)$ kite graph $(t\geq 2)$ and the path P_n are square graceful graphs.

Keywords: Square graceful graph, square graceful labeling.

AMS Subject Classification (2010): 05C69.

1 Introduction

Throughout this paper, by a graph we mean a finite, undirected, simple graph $G(V,E)$ with p vertices and q edges. A detailed survey of graph labeling can be found in [2]. There are different types of graceful labelings like odd graceful labeling, even graceful labeling and skolem - graceful labeling to various classes of graphs. In this paper, we introduce a new graceful labeling called square graceful labeling. We use the following definitions in the subsequent section.

Definition 1.1. A (p,q) graph $G(V,E)$ is said to be a square graceful graph if there exists an injection $f:V(G)\to\{0,1,2,3,...\ q^2\}$ such that the induced mapping $f_p:E(G)\to\{1,4,9,\ ..,\ q^2\}$ defined by $f_p(uv)=|f(u)-f(v)|$ is a bijection. The function f is called a square labeling of G.

Definition 1.2.[5] The corona $G_1 \Theta G_2$ of two graphs G_1 and G_2 is defined as the graph G obtained by taking one copy of G_1(which has p vertices) and p copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.

Definition 1.3. [5] A complete biparitite graph $K_{1,n}$ is called a star and it has n+1 vertices and n edges.

Definition 1.4. [5] The bistar graph $B_{m,n}$ is the graph obtained from a copy of star $K_{1,m}$ and a copy of star $K_{1,n}$ by joining the vertices of maximum degree by an edge.

Definition 1.5. [3] The graph (P_m, S_n) is obtained from m copies of the star graph S_n and the path $P_m : \{u_1, u_2, ..., u_m\}$ by joining u_j with the center of the j^{th} copy of S_n by means of an edge, for $1 \le j \le m$.

Definition 1.6. [1] A subdivision of a graph G is a graph that can be obtained from G by a sequence of edge subdivisions.

Definition 1.7. [4] An (n,t)-kite graph ,consists of a cycle of length n with a t-edge path (the tail) attached to one vertex.

In this paper, we prove that the star $K_{1,n}$, bistar $B_{m,n}$, the graph obtained by the subdivision of the edges of the star $K_{1,n}$, the graph obtained by the subdivision of the central edge of the bistar $B_{m,n}$, the generalised crown $C_3 \Theta K_{1,n}$, graph $P_m \Theta nK_1$ $(n \ge 2)$, the comb $P_n \Theta K_1$, graph (P_m, S_n), $(3,t)$ kite graph $(t \ge 2)$ and the path P_n are square graceful graphs.

2 Main Results

Theorem 2.1. The star $K_{1,n}$ is square graceful for all n.

Proof: Let $V(K_{1,n}) = \{u_i / 1 \le i \le n+1\}$. Let $E(K_{1,n}) = \{u_{n+1} u_i / 1 \le i \le n\}$.Define an injection $f : V(K_{1,n}) \to \{0,1,2,3,...,n^2\}$ by $f(u_i) = i^2$ *if* $1 \le i \le n$ and $f(u_{n+1}) = 0$. Then f induces a bijection $f_p : E(K_{1,n}) \to \{1,4,9,...,n^2\}$.

Example 2.2. A square graceful labeling of star $K_{1,12}$ is shown in Figure 1.

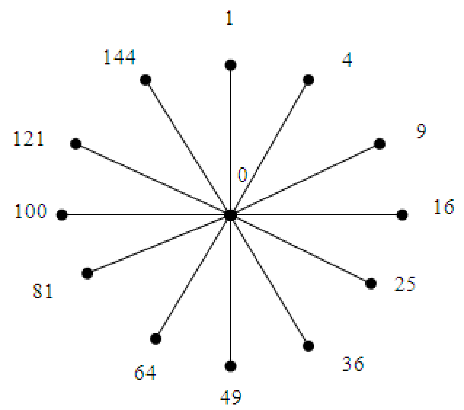

Figure 1: A *square graceful labeling of star* $K_{1,12}$.

Theorem 2.3. The graph obtained by the subdivision of the edges of the star $K_{1,n}$ is a square graceful graph.

Proof: Let G be the graph obtained by the subdivision of the edges of the star $K_{1,n}$. Let $V(G)=\{v, u_i, w_i \,/\, 1\le i\le n\}$ and $E(G)=\{v w_i, w_i u_i \,/\, 1\le i\le n\}$. Define an injection $f: V(G)\to\{0,1,2,3,...,4n^2\}$ by $f(w_i)=(2n+1-i)^2$ for $1\le i\le n$, $f(u_i)=n(3n+2-2i)$ for $1\le i\le n$, $f(v)=0$. Then, f induces a bijection $f_p: E(G)\to\{1,4,9,...,4n^2\}$ and hence the subdivision of the edges of the star $K_{1,n}$ is a square graceful graph. ■

Example 2.4. A square graceful labeling of the graph obtained by the subdivision of the edges of the star $K_{1,5}$ is shown in Figure 2.

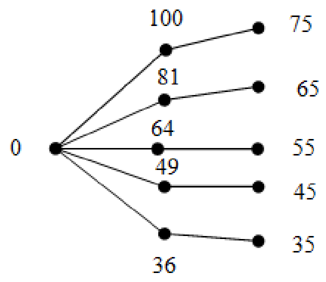

Figure 2: *A square graceful labeling of the graph obtained by the subdivision of the edges of the star* $K_{1,5}$

Theorem 2.5. Every bistar $B_{m,n}$ is a square graceful graph.

Proof: Let $B_{m,n}$ be the bistar graph with $m+n+2$ vertices. Let $V(B_{m,n})=\{u_i, v_j \,/\, 1\le i\le m+1, 1\le j\le n+1\}$ and $E(B_{m,n})=\{u_i u_{m+1}, v_j v_{n+1}, u_{m+1}v_{n+1} \,/\, 1\le i\le m, 1\le j\le n\}$.

Case (i): $m>n$.

Define an injection $f: V(B_{m,n})\to\{0,1,2,3,...,(m+n+1)^2\}$ by

$f(u_i)=(m+n+2-i)^2$ if $1\le i\le m$; $f(u_{m+1})=0$;

$f(v_j)=(n+2-j)^2+1$ if $1\le j\le n$; $f(v_{n+1})=1$.

Case (ii): $m<n$.

Define an injection $f: V(B_{m,n})\to\{0,1,2,3,...,(m+n+1)^2\}$ by

$f(u_i)=(m+2-i)^2$ *if* $1\le i\le m$; $f(u_{m+1})=1$,

$f(v_j)=(m+n+2-j)^2+1$ *if* $1\le j\le n$; $f(v_{n+1})=0$.

Case (iii): $m=n$.

Define an injection $f: V(B_{n,n})\to\{0,1,2,3,...,(2n+1)^2\}$ by

$f(u_{n+1})=0$; $f(u_i)=(2n+2-i)^2$ *if* $1\le i\le n$;

$f(v_j)=(n+2-j)^2+1$ *if* $1\le j\le n$; $f(v_{n+1})=1$.

In all the above three cases, f induces a bijection $f_p : E(B_{m.n}) \to \{1,4,9,....,(m+n+1)^2\}$. ■

Theorem 2.6. The graph obtained by the subdivision of the central edge of the bistar $B_{m,n}$ is a square graph.

Proof: Let G be the graph obtained by the subdivision of the central edge of the bistar $B_{m,n}$.

Let $V(G)=\{w, u_i, v_j / 1 \le i \le m+1, 1 \le j \le n+1\}$. Then $E(G) = \{u_i u_{m+1}, v_j v_{n+1}, \quad wu_{m+1}, wv_{n+1} /$ $1 \le i \le m, 1 \le j \le n\}$.

Case (i): $m > n$.

Define an injection $f : V(G) \to \{0,1,2,3,...,(m+n+2)^2\}$ by

$f(u_{m+1}) = (m+n+2)^2$, $f(v_{n+1}) = (m+n+1)^2$; $f(w) = 0$;

$f(u_i) = (2m+2n+3-i)(1-i)$ if $1 \le i \le m$; $f(v_j) = (2m+2n+2-j)j$ if $m+1 \le j \le m+n$.

Case (ii): $m < n$.

Define an injection $f : V(G) \to \{0,1,2,3,...,(m+n+2)^2\}$ by

$f(u_{m+1}) = (m+n+1)^2$; $f(v_{n+1}) = (m+n+2)^2$; $f(w) = 0$;

$f(u_i) = (2m+2n+2-i)i$ if $n+1 \le i \le m+n$; $f(v_j) = (2m+2n+3-j)(2+j)$ if $1 \le j \le m$.

Case (iii): $m = n$.

Define an injection $f : V(G) \to \{0,1,2,3,...,(2n+2)^2\}$ by

$f(u_{n+1}) = (2n+2)^2$; $f(v_{n+1}) = (2n+1)^2$; $f(w) = 0$;

$f(u_i) = 2(4n+3-2i)i$ *if* $1 \le i \le n$;

$f(v_j) = (4n+3-2j)(2j-1)$ *if* $1 \le j \le n$.

In all the above three cases, f induces a bijection $f_p : E(G) \to \{1,4,9,...,(m+n+2)^2\}$. ■

Example 2.7. A square graceful labeling of bistar $B_{5,3}$ is shown in Figure 3.

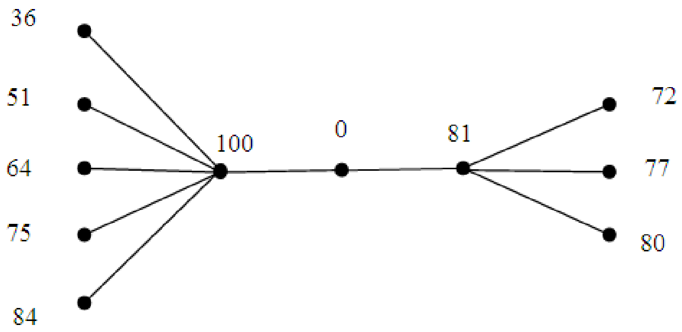

Figure 3: *A square graceful labeling of* $B_{5,3}$.

Theorem 2.8. The generalised crown $C_3 \Theta K_{1,n}$ is a square graceful graph.

Proof: Let $\{v_1, v_2, v_3, u_{i_1}, u_{i_2}, ..., u_{i_n}\}$ be the vertices of $C_3\Theta K_{1,n}$. Here $\{v_1, v_2, v_3\}$ are the vertices of C_3 and $u_{i_1}, u_{i_2}, ..., u_{i_n}$ are the vertices of the i^{th} copy of $K_{1,n}$ adjacent to v_i for $i = 1,2,3$ and the size of the graph is $q = 3n + 3$.

Define an injection $f : V(C_3\Theta K_{1,n}) \to \{0,1,2,3,...,(3n+3)^2\}$ by

$f(v_1) = 0$; $f(v_2) = 25$; $f(v_3) = 16$; $f(u_{i_j}) = (3n + 4 - j)^2$ *if* $1 \le j \le n$;

$f(u_{2_j}) = (2n + 4 - j)^2 + 25$ if $1 \le j \le n$; $f(u_{3_j}) = (n + 4 - j)^2 + 16$ if $1 \le j \le n - 2$; $f(u_{3_{(n-1)}}) = 20$ and $f(u_{3_n}) = 17$. Then f induces a bijection $f_p : E(C_3\Theta K_{1,n}) \to \{1,4,9,....,(3n+3)^2\}$ and hence the generalised crown $C_3\Theta K_{1,n}$ is a square graceful graph. ■

Example 2.9. A square graceful labeling of $C_3\Theta K_{1,4}$ is shown in Figure 4.

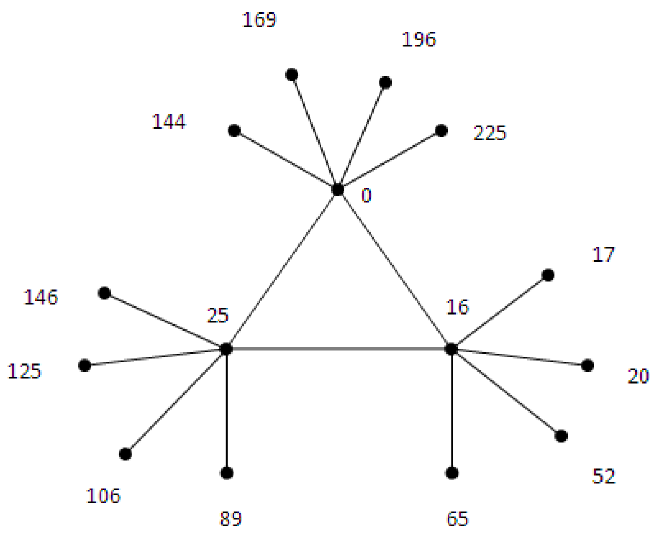

Figure 4: *A square graceful labeling of* $C_3\Theta K_{1,4}$

Theorem 2.10. The graph $P_m\Theta nK_1$ $(n \ge 2)$ is a square graceful graph.

Proof: Let $\{u_1, u_2, ..., u_m\}$ be the vertices of path p_m and $\{v_{1_j}, v_{2_j}, ..., v_{n_j}\}$ be the i^{th} copy of the null graph nK_1. Then $\{v_{1_j}, v_{2_j}, ..., v_{n_j}\}$ are the n pendent vertices adjacent to the vertex u_j of P_m for $1 \le j \le m$.

Define an injection $f : V(P_m\Theta nK_1) \to \{0,1,2,3,...,(mn + m - 1)^2\}$ by

$$f(u_j) = \frac{j(j-1)(2j-1)}{6} \text{ if } 1 \le j \le m;$$

$$f(v_{i_j}) = (q - (i-1)m - j + 1)^2 + \frac{j(j-1)(2j-1)}{6} \quad \text{if} \quad 1 \le i \le n, 1 \le j \le m.$$

Then, f induces a bijection $f_p : E(P_m \Theta nK_1) \to \{1,4,9,...,(mn+m-1)^2\}$. Hence, $P_m \Theta nK_1$ $(n \geq 2)$ is a square graceful graph. ∎

Example 2.11. A square graceful labeling of $P_5 \Theta 4K_1$ is shown in Figure 5.

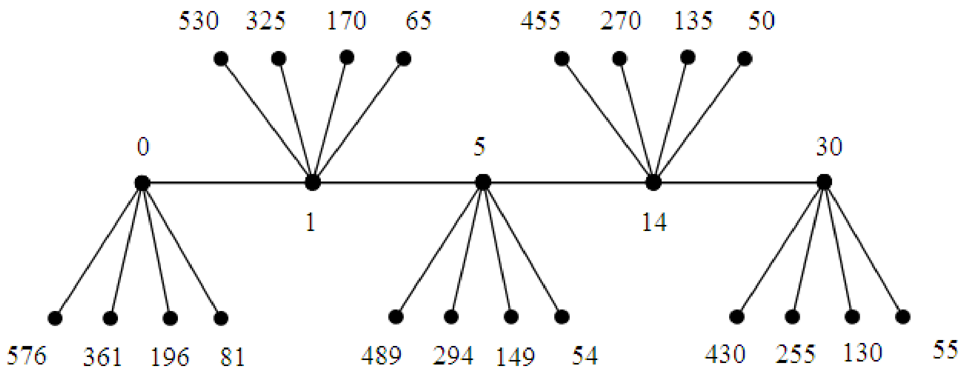

Figure 5: *A square graceful labeling of* $P_5 \Theta 4K_1$.

Corollary 2.12. The comb $P_n \Theta K_1$ is a square graceful graph.

Proof: Let $P_n \Theta K_1$ be the comb graph with $2n$ vertices. Let $V(P_n \Theta K_1) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(P_n \Theta K_1) = \{u_i u_{i+1} : 1 \leq i \leq n-1 \; ; \; u_i v_i : 1 \leq i \leq n\}$.

Define an injection $f : V(P_n \Theta K_1) \to \{0,1,2,3,....,(2n-1)^2\}$ by

$$f(u_i) = \frac{i(i-1)(2i-1)}{6} \text{ if } 1 \leq i \leq n;$$

$$f(v_i) = \frac{i(i-1)(2i-1)}{6} + (q-i+1)^2 \text{ if } 1 \leq i \leq n.$$

Then f induces a bijection $f_p : E(P_n \Theta K_1) \to \{1,4,9,....,(2n-1)^2\}$. Hence, the comb $P_n \Theta K_1$ is a square graceful graph. ∎

Example 2.13. A square graceful labeling of $P_7 \Theta K_1$ is shown in Figure 6.

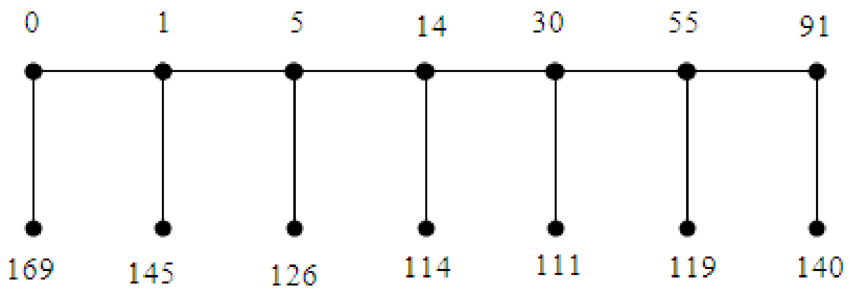

Figure 6: *A square graceful labeling of* $P_7 \Theta K_1$.

Theorem 2.14. The graph (P_n, S_m) is a square graceful graph.

Proof: Let $\{u_1, u_2, ..., u_n\}$ be the vertices of path P_n and $\{v_{o_j}, v_{1_j}, v_{2_j}, ..., v_{m_j}\}$ be the vertices of j^{th} copy P_m for $1 \le j \le n$.

Then $E((P_n, S_m)) = \begin{cases} u_i\, u_{i+1} & if\ 1 \le i \le n-1 \\ u_i\, v_{0_i} & if\ 1 \le i \le n \\ v_{o_j}\, v_{i_j} & if\ 1 \le i \le m,\ 1 \le j \le n \end{cases}$

Define an injection $f : V((P_n, S_m)) \to \{0,1,2,3,....,(mn+2n-1)^2\}$ by

$f(u_i) = \dfrac{i(i-1)(2i-1)}{6}$ $if\ 1 \le i \le n$;

$(\ f(v_{0_j}) = (q-j+1)^2 + \dfrac{j(j-1)(2j-1)}{6}$ if $1 \le j \le n$;

$f(v_{i_j}) = (2q - ni - 2j + 2)(ni) + \dfrac{j(j-1)(2j-1)}{6}$ if $1 \le i \le m, 1 \le j \le n$.

Then, f induces a bijection $f_p : E(P_n, S_m) \to \{1,4,9,....,(mn+2n-1)^2\}$ and hence (P_n, S_m) is square graceful. ■

Example 2.15. A square graceful labeling of (P_5, S_3) is shown in Figure 7.

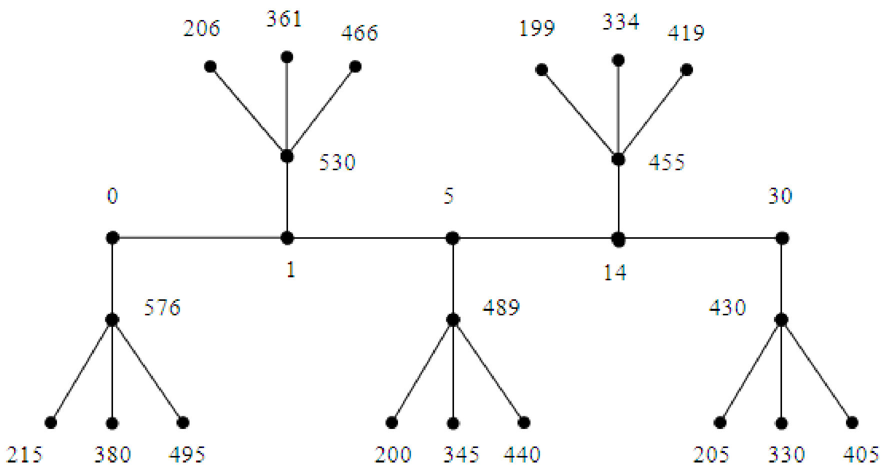

Figure 7: *A square graceful labeling of* (P_5, S_3).

Theorem 2.16. $(3,t)$ kite graph is square graceful for $t \ge 2$.

Proof: Let $\{v_1, v_2, v_3\}$ be the three vertices of a cycle C_3 and $\{u_1, u_2, ..., u_t\}$ be the t vertices of the tail with v_1 adjacent to u_1. Therefore, the size of G is $q = 3 + t$. We prove the theorem in two cases.

Case (i): $t = 2$.

Define a bijection $f : V(G) \to \{0,1,2,3,...,(3+t)^2\}$ as follows.

$f(v_1)=0$, $f(v_2)=16$, $f(v_3)=25$, $f(u_1)=4$ and $f(u_2)=3$.

Case (ii): $t>2$.

Define a bijection $f:V(G)\rightarrow\{0,1,2,3,...,(3+t)^2\}$ as follows.

$$f(u_i)=\sum_{j=1}^{i}(-1)^{i+1}\,[q-j+1]^2 \;\; if \;\; 1\le i\le q-5;$$

$$f(u_{t-1})=\sum_{j=1}^{q-5}(-1)^{j+1}\,[q-j+1]^2-4;$$

$$f(u_t)=\sum_{j=1}^{q-5}(-1)^{j+1}\,[q-j+1]^2-5; \quad f(v_1)=0\;,\; f(v_2)=16\;,\; f(v_3)=25.$$

In both the cases f induces a bijection $f_p:E(G)\rightarrow\{1,4,9,.....,(3+t)^2\}$ and hence $(3,t)$ kite graph is square graceful for $t\ge 2$. ∎

Example 2.17. A square graceful labeling of $(3,3)$ kite is shown in Figure 8.

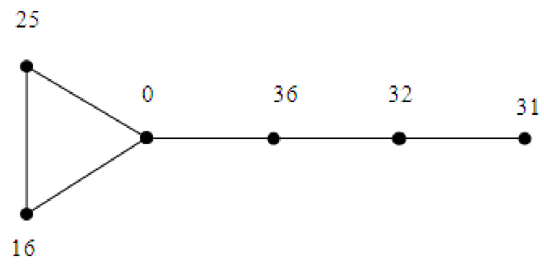

Figure 8: *A square graceful labeling of* $(3,3)$ *kite.*

Theorem 2.18. Every path P_n is a square graceful graph.

Proof: Let P_n be a path graph with n vertices $\{u_1,u_2,....,u_n\}$. Let $E(P_n)=\{u_iu_{i+1} \,/\, 1\le i\le n-1\}$.

Define an injection $f:V((P_n)\rightarrow\{0,1,2,3,.....,(n-1)^2\}$ by

$f(u_1)=0$ and $f(u_{i+1})=\sum_{j=1}^{i}(-1)^{j+1}(n-j)^2$ for $1\le i\le n-1$. Then f induces a bijection $f_p:E(P_n)\rightarrow\{1,4,9,.....,(n-1)^2\}$. Hence, every path P_n is a square graceful graph. ∎

References

[1] Diari Indriati, *On* γ*-Labeling of* (*n*,*t*)*-Kite Graph*, Jurnal Matematika & Sains, Vol.16, No.3(2011), 129-132.

[2] J. A. Gallian, *A dynamic survey of graph labeling*, The Electronic journal of Combinatorics,16(2013), # DS6.

[3] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Massachusetts, 1972.

[4] K. Murugan and A. Subramanian, *Labeling of Subdivided Graphs*, American J. of Mathematics and Sciences, Vol.1, No.1(2012), 143-149.

[5] Selvam Avadayappan and R. Vasuki, *New Families of Mean Graphs*, International J. Math. Combin, Vol.2 (2010), 68-80.