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Many more families of mean graphs

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Abstract

For every assignment $f: V(G) \rightarrow \{0, 1, 2, \ldots, q\}$, an induced edge labeling $f^* : E(G) \rightarrow \{1, 2, 3, \ldots, q\}$ is defined by $\frac{f(u)+f(v)}{2}$ if $f(u)$ and $f(u)$ are of same parity and by $\frac{f(u)+f(v)+1}{2}$ otherwise for every edge $uv \in E(G)$. If $f^*(E) = \{1, 2, 3, \ldots, q\}$, then we say that f is a mean labeling of G. If a graph G admits a mean labeling, then G is called a mean graph. In this paper, we prove that the graphs $C_n+v_1v_3$ $(n\geq4)$, $C_2(P_n)$, $n\geq2$, $T_n(C_m)$, $n \geq 2$, $m\geq3$, $DQ(n)$, $n \geq 2$, $TQ(n)$, $n \geq 2$ and mC_n – snake, $m \geq 1$, $n \geq 3$ are mean graphs..

Keywords: Labeling, mean labeling, mean graph.

AMS Subject Classification (2010): 05C78.

1 Introduction

Throughout this paper, by a graph we mean a finite, undirected, simple graph. Let $G(V,E)$ be a graph with p vertices and q edges. For notations and terminology, we follow [1]. *Path* on n vertices is denoted by P_n and a *cycle* on n vertices is denoted by C_n. A *triangular snake* T_n is obtained from a path $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+1} to a new vertex u_i for $1 \leq i \leq n-1$, that is, every edge of a path is replaced by a triangle C_3. The graph T_6 is shown in Figure 1.

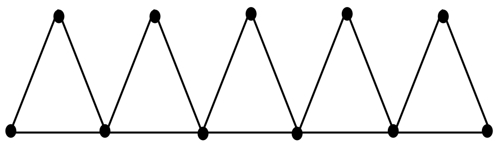


Figure 1: Triangular snake T_6.

Let $Q(n)$ be the quadrilateral snake obtained from the path $v_1, v_2, v_3, \ldots, v_n$ by joining v_i and v_{i+1} to new vertices u_i and w_i. That is, every edge of a path is replaced by a cycle C_4. The quadrilateral snake $Q(3)$ is given in Figure 2.

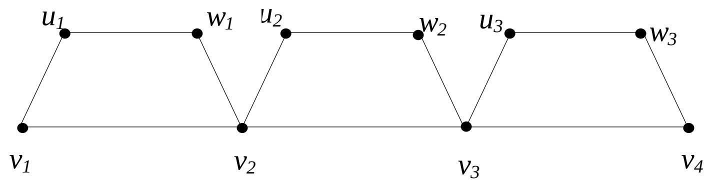
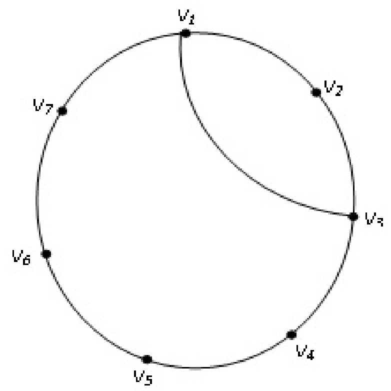


Figure 2: Quadrilateral snake $Q(3)$.

The graph $C_n+v_1v_3$ is obtained from the cycle C_n: $v_1v_2 \ldots v_nv_1$ by adding an edge between the vertices v_1 and v_3. An example for the graph $C_7+v_1v_3$ is shown in Figure 3.

Let T_n be the triangular snake obtained from the path P_n: $v_1v_2 \ldots v_n$. Then the double triangular snake $C_2(P_n)$ is obtained from T_n by adding new vertices $w_1, w_2, \ldots, w_{n-1}$ and edges v_iw_i and w_iv_{i+1} for $1 \leq i \leq n-1$. The graph $C_2(P_5)$ is given in Figure 4.



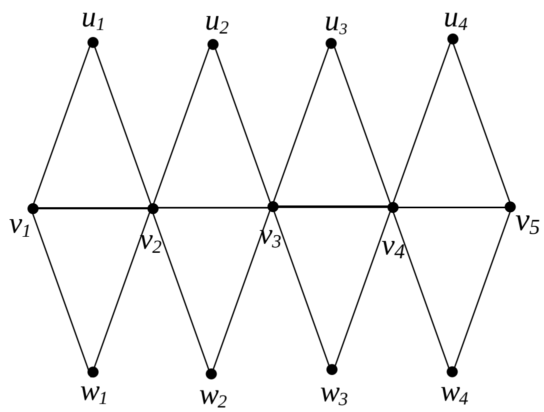


Figure 3: $C_7+v_1v_3$.

Figure 4: *Double triangular snake* $C_2(P_5)$.

The balloon of the triangular snake $T_n(C_m)$ is the graph obtained from C_m by identifying an end vertex of the basic path in T_n at a vertex of C_m. The balloon graph $T_5(C_6)$ is given in Figure 5.

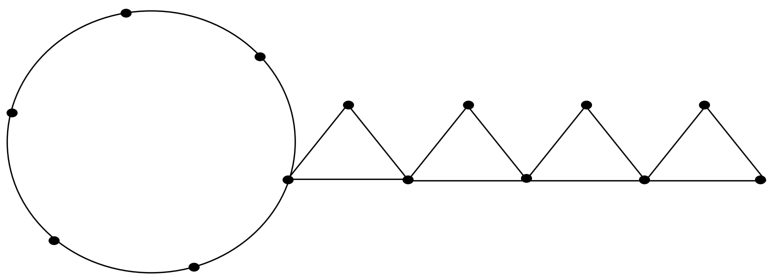


Figure 5: *The balloon of the triangular snake* $T_5(C_6)$.

Let $Q(n)$ be the quadrilateral *snake* obtained from the path $v_1, v_2, v_3, \ldots, v_n$. Then the double quadrilateral snake $DQ(n)$ is obtained from $Q(n)$ by adding new vertices $s_1, s_2, s_3, \ldots, s_{n-1}$; $t_1, t_2, t_3, \ldots, t_{n-1}$ and new edges $v_i s_i$, $t_i v_{i+1}$, $s_i t_i$ for $1 \leq i \leq n-1$. The graph $DQ(3)$ is shown in Figure 6.

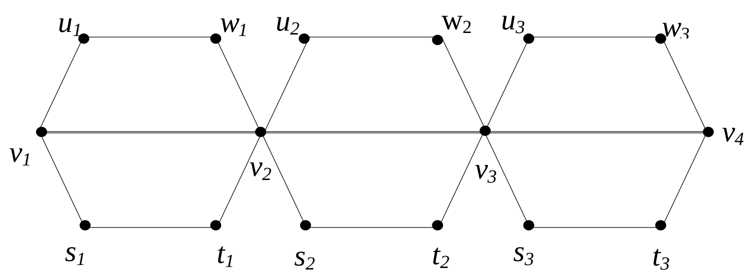


Figure 6: *Double quadrilateral snake DQ(n).*

Let D$Q(n)$ be the double quadrilateral snake obtained from the quadrilateral snake $Q(n)$ by adding new vertices s_i and t_i. Then the triple quadrilateral snake $TQ(n)$ is obtained from D$Q(n)$ by adding new vertices $x_1, x_2, x_3, \ldots, x_{n-1}$; $y_1, y_2, y_3, \ldots, y_{n-1}$ and new edges v_ix_i, y_iv_{i+1}, x_iy_i for $1 \leq i \leq n-1$. For example, the graph $TQ(2)$ is given in Figure 7.

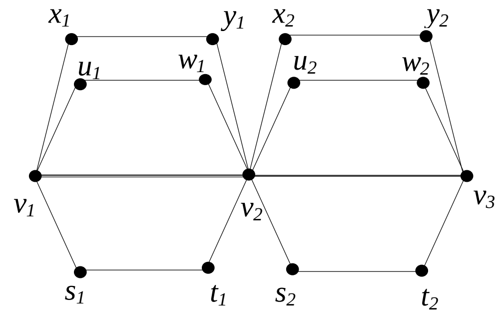


Figure 7: *Triple quadrilateral snake TQ(n).*

A cyclic snake mC_n is the graph obtained from m copies of C_n by identifying the vertex $v_{(k+2)_j}$ in the j^{th} copy at a vertex $v_{1_{j+1}}$ in the $(j+1)^{th}$ copy if $n = 2k + 1$ and identifying the vertex $v_{(k+1)_j}$ in the j^{th} copy at a vertex $v_{1_{j+1}}$ in the $(j+1)^{th}$ copy if $n = 2k$. The cycle snake graph $3C_6$ is shown in Figure 8.

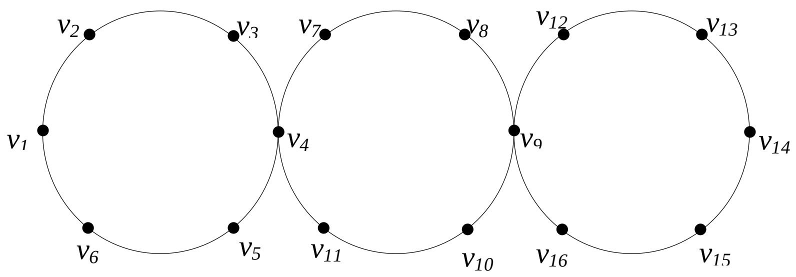


Figure 8: Cyclic snake graph $3C_6$.

A graph labeling is an assignment of integers or a subset of a set to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or an edge labeling). A vertex labeling f is called a mean labeling of G if its induced edge labeling $f^*: E(G) \to \{1, 2, \cdots, q\}$ defined by

$$f^*(uv) = \begin{cases} \dfrac{f(u) + f(v)}{2} & \text{if } f(u) \text{ and } f(v) \text{ are of same parity} \\ \dfrac{f(u) + f(v) + 1}{2} & \text{otherwise.} \end{cases}$$

is a bijection. We say that f is a mean labeling of G. If a graph G has a mean labeling, then we say that G is a mean graph.

The concept of mean labeling was first introduced by S. Somasundaram and R. Ponraj [8] in 2003. The meanness of many standard graphs like P_n, C_n, $K_n (n\leq 3)$, the ladder, the triangular snake, $K_{1,2}$, $K_{1,3}$, $K_{2,n}$, K_2+mK_1, $K_n^c+2K_2$, S_m+K_1, $C_m \cup P_n (m\geq 3,\ n\geq 2)$, quadrilateral snake, comb, bistars $B(n)$, $B_{n+1,n}$, $B_{n+2,n}$, the carona of a ladder, subdivision of the central edge of $B_{n,n}$, subdivision of the star $K_{1,n}$, the friendship graph $C_3^{(2)}$, the crown $C_n \odot K_1$, $C_n^{(2)}$, the dragon, arbitrary super subdivision of a path are proved in [8], [9], [10], [11], [2], [3]. In addition, they have proved that the graphs $K_n(n>3)$, $K_{1,n}(n>3)$, $B_{m,n}(m>n+2)$, $S(K_{1,n})$, $n>4$, $C_3^{(t)}$ $(t>2)$ and the wheel W_n are not mean graphs. In [4], the meanness of the following graphs have been proved: $C_m\times P_n$; the caterpillar $P(n,2,3)$; $Q_3 \times P_{2n}$; corona of a H – graph; mC_3; $C_n \cup K_{1,m}$ $(n\geq 3,\ 1\leq m\leq 4)$; $mC_3 \cup K_{1,m}$ $(1\leq m\leq 4)$; the dragon $P_n(C_m)$ and some standard graphs. In [5], the meanness of the graphs $(P_m;C_n)$, $m\geq 1$, $n\geq 3$, $(P_m;Q_3)$, $m\geq 1$, $(P_{2n};S_m)$, $m\geq 3$, $n>1$, $(P_n;S_1)$, $(P_n;S_2)$, $n\geq 1$ have been proved. The meanness of the following product related graphs (P_3; $C_3\times K_2$), $G\times K_2$ for any mean graph G with $p = q+1$ and the train graph $P_k(G,u,v)$ where G is a mean graph have been proved in [6]. It is also proved that $G^k(u,v)$ is a mean graph where G is a mean graph with two vertices u and v such that $f(u) = 0$ and $f(v) = q$ in [7].

In this paper, we prove the meanness of the graphs $C_n+v_1v_3$ $(n\geq 4)$, $C_2(P_n)$, $n\geq 2$, $T_n(C_m)$, $n \geq 2$, $m\geq 3$, $DQ(n)$, $n \geq 2$, $TQ(n)$, $n \geq 2$ and mC_n – snake, $m \geq 1$, $n \geq 3$.

2 Main Results

Theorem 2.1. $C_n+v_1v_3$ is a mean graph for $n \geq 4$.

Proof: Let C_n be a cycle with vertices $v_1, v_2, v_3, \ldots, v_n$ and edges $e_1, e_2, e_3, \ldots, e_n$, where $e_i = v_iv_{i+1}$ where '+' is addition modulo n.

Define $f : V(C_n + v_1v_3) \rightarrow \{0, 1, 2, \ldots, n+1\}$ as follows:

Case 1: When n is odd, $n = 2m+1$, $m = 2, 3, 4, \ldots$

$f(v_1) = 0$; $f(v_2) = 2$;

$f(v_i) = 2i-1$, $3\leq i \leq m+1$; $f(v_{m+j+1}) = n-2j+3$, $1 \leq j \leq m$.

Then the induced edge labels are

$f^*(e_1) = 1$; $f^*(e_i) = 2i$, $2 \leq i \leq m+1$;

$f^*(e_{m+j+1}) = n-2j+2$, $1 \leq j \leq m-1$; $f^*(e_{2m+1}) = 2$; $f^*(v_1v_3) = 3$.

Case 2: When n is even, $n = 2m$, $m = 2, 3, 4, \ldots$

$f(v_1) = 0$; $f(v_2) = 2$;

$f(v_i) = 2i-1$, $3 \leq i \leq m+1$; $f(v_{m+j+1}) = n-2j+2$, $1 \leq j \leq m-1$.

Then the induced edge labels are

$f^*(e_1) = 1;$ $f^*(e_i) = 2i,\ 2 \le i \le m;$

$f^*(e_{m+j}) = n\text{-}2j+3,\ 1 \le j \le m\text{-}1;$ $f^*(e_{2m}) = 2;$ $f^*(v_1v_3) = 3.$

Clearly, f is a mean labeling of $C_n+v_1v_3$. ■

A mean labelings of the graphs $C_7+v_1v_3$ and $C_{10}+v_1v_3$ are shown in Figure 9.

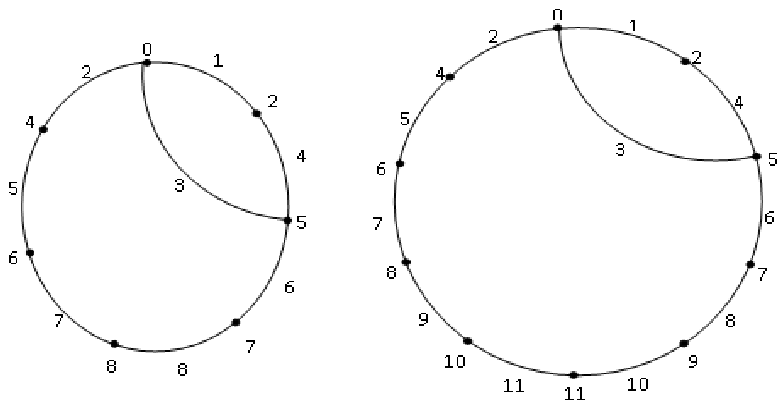


Figure 9: *Mean labelings of* $C_7+v_1v_3$ *and* $C_{10}+v_1v_3$.

Theorem 2.2. $C_2(P_n)$ is a mean graph for $n \ge 2$.

Proof: Let $v_1, v_2, v_3, \ldots, v_n; u_1, u_2, u_3, \ldots, u_{n-1}$ and $w_1, w_2, w_3, \ldots, w_{n-1}$ be the vertices of $C_2(P_n)$. Define $f : V(C_2(P_n)) \to \{0, 1, 2, \ldots, 5n-5\}$ as follows:

$f(v_i) = 5(i\text{-}1),\ 1 \le i \le n;$ $f(u_i) = 5i\text{-}1,\ 1 \le i \le n\text{-}1;$

$f(w_i) = 5i\text{-}3,\ 1 \le i \le n\text{-}1.$

Then the induced edge labels are

$f^*(v_iv_{i+1}) = 5i\text{-}2,\ 1 \le i \le n\text{-}1;$ $f^*(v_iu_i) = 5i\text{-}3,\ 1 \le i \le n\text{-}1;$

$f^*(v_iw_i) = 5i\text{-}4,\ 1 \le i \le n\text{-}1;$ $f^*(u_iv_{i+1}) = 5i,\ 1 \le i \le n\text{-}1;$

$f^*(w_iv_{i+1}) = 5i\text{-}1,\ 1 \le i \le n\text{-}1.$

Clearly f is a mean labeling of $C_2(P_n)$. ■

A mean labeling of the graph $C_2(P_5)$ is illustrated in Figure 10.

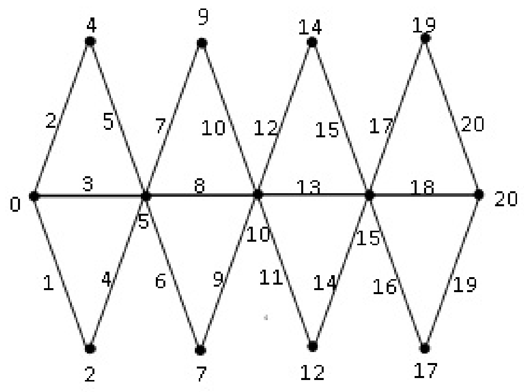


Figure 10: *A mean labeling of the graph* $C_2(P_5)$.

Theorem 2.3. $T_n(C_m)$ is a mean graph for $n \geq 2$, $m \geq 3$.

Proof: Let $v_1, v_2, v_3, \ldots, v_m$ be the vertices of C_m and $u_1, u_2, u_3, \ldots, u_n$; $w_1, w_2, w_3, \ldots, w_{n-1}$ be the vertices of T_n.
Then define g on $T_n(C_m)$ as follows:

Case 1: when m is even, $m = 2k$, $k = 2, 3, 4, \ldots$

$g(v_i) = f(v_i)$, $1 \leq i \leq m$;
$g(u_i) = m+3i-3$, $1 \leq i \leq n$;
$g(w_i) = m+3i-1$, $1 \leq i \leq n-1$.

Then the induced edge labels are

$g^*(e_i) = f(e_i)$, $1 \leq i \leq m$; $g^*(u_iu_{i+1}) = m+3i-1$, $1 \leq i \leq n-1$;
$g^*(u_iw_i) = m+3i-2$, $1 \leq i \leq n-1$; $g^*(w_iu_{i+1}) = m+3i$, $1 \leq i \leq n-1$.

Case 2: when m is odd, $m = 2k + 1$, $k = 1, 2, 3, \ldots$

$g(v_i) = f(v_i)$, $1 \leq i \leq m$;
$g(u_i) = m+3i-3$, $1 \leq i \leq n$;
$g(w_i) = m+3i-1$, $1 \leq i \leq n-1$.

Then the induced edge labels are

$g^*(e_i) = f(e_i)$, $1 \leq i \leq m$; $g^*(u_iu_{i+1}) = m+3i-1$, $1 \leq i \leq n-1$;
$g^*(u_iw_i) = m+3i-2$, $1 \leq i \leq n-1$; $g^*(w_iu_{i+1}) = m+3i$, $1 \leq i \leq n-1$.

Clearly g is a mean labeling of $T_n(C_m)$. ■

A mean labelings of the graphs $T_5(C_6)$ and $T_5(C_9)$ are given in Figure 11(a) and 11(b) respectively.

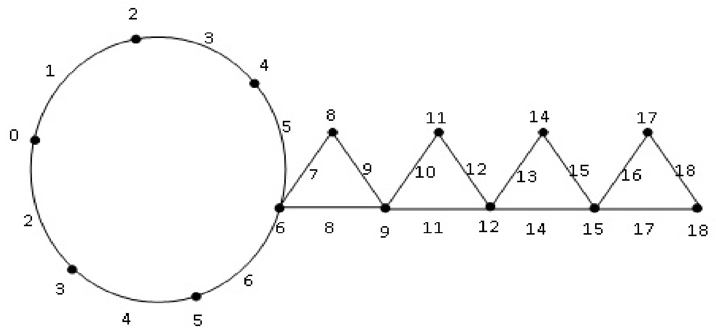


Figure 11(a): *A mean labeling of* $T_5(C_6)$.

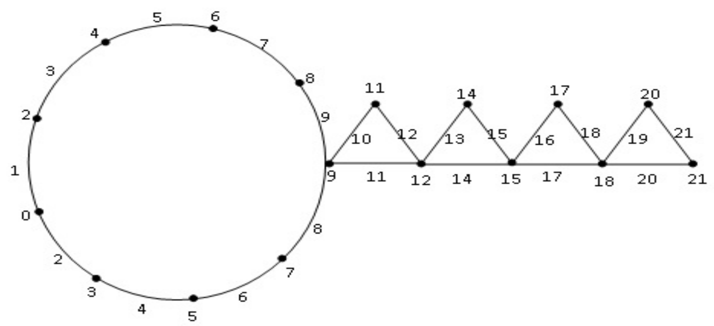


Figure 11(b): *A mean labeling of* $T_5(C_9)$.

Theorem 2.4. The double quadrilateral snake *DQ(n)* is a mean graph for $n \geq 2$.

Proof: Let $v_1, v_2, v_3, \ldots, v_n; u_1, u_2, u_3, \ldots, u_{n-1}; w_1, w_2, w_3, \ldots, w_{n-1}; s_1, s_2, s_3, \ldots, s_{n-1}; t_1, t_2, t_3, \ldots, t_{n-1}$ be the vertices of *DQ(n)*.

Define $f : V(DQ(n)) \to \{0,1,2,\ldots,7n-7\}$ as follows:

$f(v_i) = 7(i-1),\ 1 \leq i \leq n;$ $\quad f(u_i) = 7i-5,\ 1 \leq i \leq n\text{-}1;$

$f(w_i) = 7i-3,\ 1 \leq i \leq n\text{-}1;$ $\quad f(s_i) = 7i-4,\ 1 \leq i \leq n\text{-}1;$

$f(t_i) = 7i-1,\ 1 \leq i \leq n\text{-}1.$

Then the induced edge labels are

$f(v_iv_{i+1}) = 7i-3,\ 1 \leq i \leq n\text{-}1;$ $\quad f(v_iu_i) = 7i-6,\ 1 \leq i \leq n\text{-}1;$

$f(w_i\, v_{i+1}) = 7i-1,\ 1 \leq i \leq n\text{-}1;$ $\quad f(u_iw_i) = 7i-4,\ 1 \leq i \leq n\text{-}1;$

$f(v_is_i) = 7i-5,\ 1 \leq i \leq n\text{-}1;$ $\quad f(t_i\, v_{i+1}) = 7i,\ 1 \leq i \leq n\text{-}1;$

$f(s_it_i) = 7i-2,\ 1 \leq i \leq n\text{-}1.$

Clearly f is a mean labeling of *DQ(n)*. ■

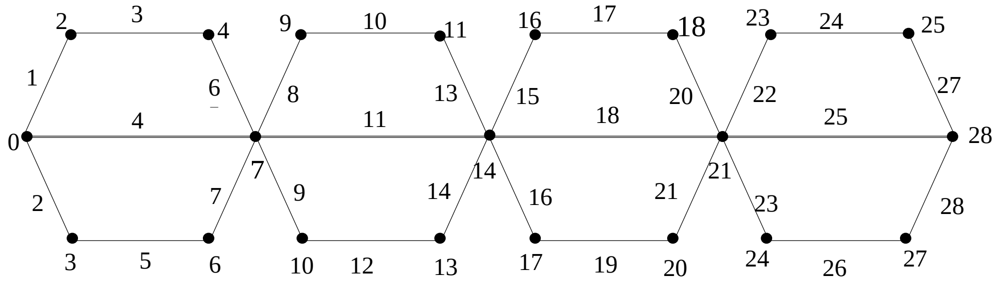


Figure 12: *A mean labeling of DQ*(4).

Theorem 2.5. The triple quadrilateral snake *TQ(n)* is a mean graph for $n \geq 2$.

Proof: Let $v_1, v_2, v_3, \ldots, v_n; u_1, u_2, u_3, \ldots, u_{n-1}; w_1, w_2, w_3, \ldots, w_{n-1}; s_1, s_2, s_3, \ldots, s_{n-1}; t_1, t_2, t_3, \ldots, t_{n-1}; x_1, x_2, x_3, \ldots, x_{n-1}; y_1, y_2, y_3, \ldots, y_{n-1}$ be the vertices of *TQ(n)*.

Define $f : V(TQ(n)) \to \{0,1,2,\ldots,10n-10\}$ as follows:

$f(v_i) = 10(i-1),\ 1 \leq i \leq n;$ $\quad f(u_i) = 10i-8,\ 1 \leq i \leq n\text{-}1;$

$f(w_i) = 10i-4,\ 1 \leq i \leq n\text{-}1;$ $\quad f(s_i) = 10i-6,\ 1 \leq i \leq n\text{-}1;$

$f(t_i) = 10i-1,\ 1 \leq i \leq n\text{-}1;$ $\quad f(x_i) = 10i-5,\ 1 \leq i \leq n\text{-}1;$

$f(y_i) = 10i-3,\ 1 \leq i \leq n\text{-}1.$

Then the induced edge labels are

$f(v_iv_{i+1}) = 10i-5,\ 1 \leq i \leq n\text{-}1;$ $\quad f(v_iu_i) = 10i-9,\ 1 \leq i \leq n\text{-}1;$

$f(w_i\, v_{i+1}) = 10i-2,\ 1 \leq i \leq n\text{-}1;$ $\quad f(u_iw_i) = 10i-6,\ 1 \leq i \leq n\text{-}1;$

$f(v_is_i) = 10i-8,\ 1 \leq i \leq n\text{-}1;$ $\quad f(t_i\, v_{i+1}) = 10i,\ 1 \leq i \leq n\text{-}1;$

$f(s_it_i) = 10i-3,\ 1 \leq i \leq n\text{-}1;$ $\quad f(v_i\, x_i) = 10i-7,\ 1 \leq i \leq n\text{-}1;$

$f(y_i\, v_{i+1}) = 10i-1,\ 1 \leq i \leq n\text{-}1;$ $\quad f(x_i\, y_i) = 10i-4,\ 1 \leq i \leq n\text{-}1.$

Clearly f is a mean labeling of *TQ(n)*. ■

A mean labeling of the graph *TQ(3)* is given in Figure 13.

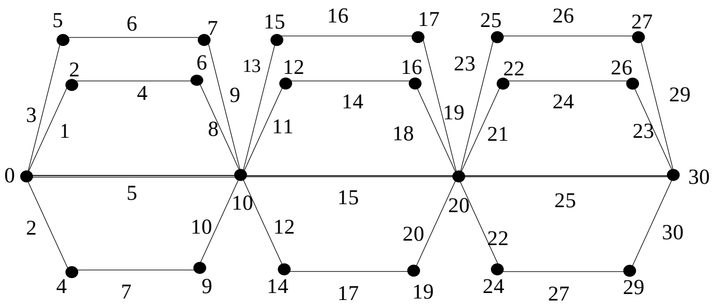


Figure 13: *A mean labeling of TQ*(3).

Theorem 2.6. The graph mC_n – snake, $m \geq 1$, $n \geq 3$ has a mean graph.

Proof: We prove this result by induction on *m*. Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_n$ be the edges of mC_n for $1 \leq j \leq m$. Let *f* be a mean labeling of the cycle C_n.

When $m = 1$, C_n is a mean graph, $n \geq 3$. Hence the result is true when $m = 1$.

Let $m = 2$. The cyclic snake $2C_n$ is the graph obtained from 2 copies of C_n by identifying the vertex $v_{(k+2)_1}$ in the first copy of C_n at a vertex v_{1_2} in the second copy of C_n when $n = 2k+1$ and identifying the vertex $v_{(k+1)_1}$ in the first copy of C_n at a vertex v_{1_2} in the second copy of C_n when $n = 2k$. Define the mean labeling *g* of $2C_n$ as follows:

For $1 \leq i \leq n$, $g(v_{i_1}) = f(v_{i_1})$, $g(v_{i_2}) = f(v_{i_1}) + n$, $g^*(e_{i_1}) = f^*(e_{i_1})$, $g^*(e_{i_2}) = f^*(e_{i_1}) + n$.

Thus, $2C_n$-snake is a mean graph.

Assume that mC_n-snake is a mean graph for any $m \geq 1$. We prove that $(m+1)C_n$-snake is a mean graph.

Let *f* be a mean labeling of mC_n. We define the mean labeling *g* on $(m+1)C_n$ as follows:

$$g(v_{i_j}) = f(v_{i_1}) + (j-1)n,\ 1 \leq i \leq n,\ 2 \leq j \leq m; \qquad g(v_{i_{m+1}}) = f(v_{i_1}) + mn,\ 1 \leq i \leq n.$$

For the vertex labeling *g*, the induced edge labeling g^* is defined as follows:

$$g^*(e_{i_j}) = f^*(e_{i_1}) + (j-1)n,\ 1 \leq i \leq n,\ 2 \leq j \leq m; \quad g^*(e_{i_{m+1}}) = f^*(e_{i_1}) + mn,\ 1 \leq i \leq n.$$

Then it can be easily verified that *g* is a mean labeling of $(m+1)C_n$- snake. ■

Mean labelings of $5C_6$-snake and $4C_7$-snake are shown in Figure 14.

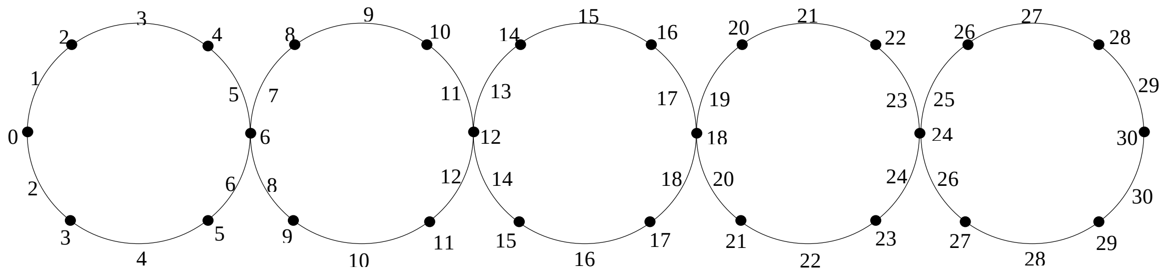


Figure 14(a): *Mean labelings of* $5C_6$.

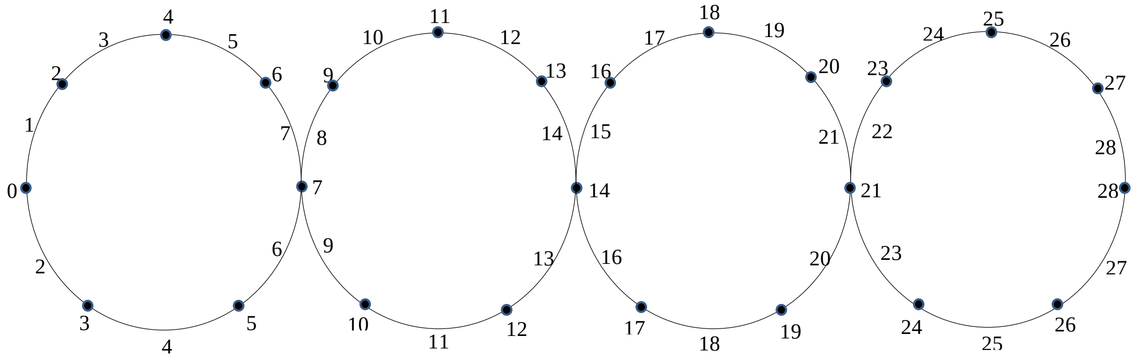


Figure 14(b)***:*** *Mean labelings of* $4C_7$*.*

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