International Journal of Mathematics and Soft Computing
Vol.4, No.1. (2014), 113 - 118.

ISSN Print : 2249 – 3328
ISSN Online: 2319 – 5215

Strong independence and strong vertex covering in semigraphs

D. K. Thakkar
Department of Mathematics, Saurashtra University Campus,
Rajkot – 360 005, INDIA.
Email: dkthakkar1@yahoo.co.in

A. A. Prajapati
Mathematics Department, L. D. College of Engineering,
Ahmedabad – 380 015, INDIA.
Email: ashishprajapati14@gmail.com

Abstract

In this paper we study the effect of removing a vertex from the semigraph on strong vertex covering number and strong independence number. Also we prove that the strong vertex covering number does not increase when a vertex is removed from the semigraph.

Keywords: Semigraph, strong independence number, strong vertex covering number.

AMS Subject Classification (2010): 05C99, 05C69, 05C07.

1 Introduction

Semigraphs provide a generalization of graphs with many applications and scope for further research. As a result, many new theorems have appeared. Semigraphs and their applications have been studied in [3]. Some authors have defined parameters like domination number, independence number in semigraph. Our objective is to study the effect of removing a vertex from a semigraph on two parameters namely strong vertex covering number and strong independence number of a semigraph. These concepts have been defined in [5]. Also we prove that the strong vertex covering number does not increase when a vertex is removed. Further we prove the corresponding theorem for strong independence number of a semigraph.

2 Preliminaries

Definition 2.1. [5] Let G be a semigraph and $S \subseteq V(G)$. Then S is said to be a strong vertex covering set if whenever x and y are adjacent in G then $x \in S$ or $y \in S$.

Definition 2.2. [5] A strong vertex set with minimum cardinality is called minimum strong vertex covering set which also can be called as $\alpha_s -$set of G .

Definition 2.3. [5] The cardinality of an $\alpha_s - set$ is called the strong vertex covering number of the semigraph G , and is denoted as $\alpha_s(G)$.

Definition 2.4. [5] If G is a semigraph and $S \subset V(G)$ then S is called a strong independent set of G, if whenever x and y belong to S and $x \neq y$, then they are non-adjacent in G.

Definition 2.5. [5] Cardinality of a maximum strong independent set of a semigraph G is called the strong independence number of G and is denoted as $\beta_s(G)$.

It is obvious that a set S is strongly independent if and only if $V(G) - S$ is a strong vertex covering set of G. $N(v)$ denotes the set of vertices, which are adjacent to v.

Consider a semigraph G and $v \in V(G)$. $G - v$ is a semigraph whose vertex set is $V(G) - \{v\}$ and edge set is $E(G - v) = \{E \in E(G) : v \notin E\}$. Also note that a set S is a maximum strong independent set of G if and only if $V(G) - S$ is a minimum strong vertex covering set of G. Hence, $\alpha_s(G) + \beta_s(G) = n$ = number of vertices in semigraph *G*.

3 Main Results

Consider a semigraph G and $v \in V(G)$. We consider the subsemigraph whose vertex set is $V(G) - \{v\}$ and the edge set is the set of all edges of G which do not contain the vertex v. We prove that the strong vertex covering number does not increase when a vertex is removed from a semigraph.

Theorem 3.1. Let G be a semigraph and $v \in V(G)$ th**en** $\alpha_s(G - v) \leq \alpha_s(G)$.

Proof: Let S be a minimum strong vertex covering set of G.

Case 1: Suppose $v \notin S$.

If x and y are adjacent vertices of $G - v$, then x and y are adjacent vertices of G. Since S is a strong vertex covering set of G, $x \in S$ or $y \in S$. Thus, S is a strong vertex covering set of $G - v$. Hence, $\alpha_s(G - v) \leq |S| = \alpha_s(G)$.

Case 2: Suppose $v \in S$.

Let $S_1 = S - \{v\}$, then S_1 is a subset of $V(G - v)$. Let x and y be adjacent vertices of $G - v$. Then x and y be adjacent vertices of G. Since S is strong vertex covering set of G, $x \in S$ or $y \in S$. Since $v \notin \{x, y\}, x \in S_1$ or $y \in S_1$. Therefore, S_1 is a strong vertex covering set of $G - v$. Thus, $\alpha_s(G - v) \leq |S_1| < |S| = \alpha_s(G)$. ■

Theorem 3.2. Let G be a semigraph and $v \in V(G)$. If there is an $\alpha_s - set$ S such that $v \in S$ then $\alpha_s(G - v) < \alpha_s(G)$.

Proof: Consider the set $S_1 = S - \{v\}$. We prove that S_1 is a strong vertex covering set of $G - v$. For this, we suppose that x and y are vertices of $G - v$ which are adjacent in $G - v$. So, there is an edge E of $G - v$ such that $x, y \in E$. Since E is also and edge of G, it follows that x and y are adjacent in G. Since S is a strong vertex covering set of G we have $x \in S$ or $y \in S$. Since $x \neq v$ and $y \neq v$, $x \in S_1$ or $y \in S_1$. Thus, S_1 is a strong vertex covering set of $G - v$. Thus, $\alpha_s(G-v) \leq |S_1| < |S| = \alpha_s(G)$ and the theorem is proved. ∎

Remarks: The above theorem says that if $v \in S$ where S is a minimum strong vertex covering set of G, then $\alpha_s(G-v) < \alpha_s(G)$. However, the above condition is not necessary.

Example 3.3. Consider the semigraph G whose vertex set is $V(G) = \{1,2,3,4,5\}$ and edge set is $E(G) = \{(1,2,3,4),(3,5),(4,5)\}$. Note that the set $S = \{2,3,4\}$ is a minimum strong vertex covering set of G .Hence $\alpha_s(G) = 3$.

Now consider the semigraph $G-1$. In this semigraph the edges are $(3,5)$ and (4,5). In this semigraph $\{5\}$ is a minimum strong vertex covering set of $G-1$. Hence, $\alpha_s(G-1) = 1$. Thus, $\alpha_s(G-v) < \alpha_s(G)$. Note that $1 \notin S$ and there is no $\alpha_s - set$ of G which contains 1.

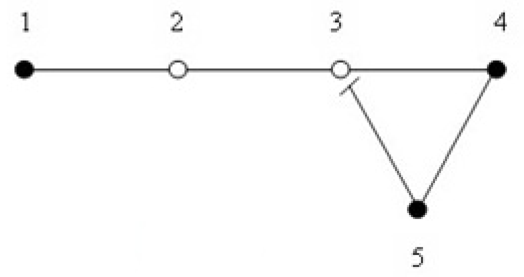

Figure 1

For a semigraph G we introduce the following notations.

$V_{cr}^0 = \{v \in V(G) : \alpha_s(G-v) = \alpha_s(G)\}$ and $V_{cr}^- = \{v \in V(G) : \alpha_s(G-v) < \alpha_s(G)\}$.

Accordingly if S is a minimum strong vertex covering set of G then for every vertex v in S, $v \in V_{cr}^-$ (From Theorem 3.2). Thus, if $S_1, S_2, ..., S_k$ are all minimum strong vertex covering sets of G, then $\bigcup S_i (i = 1,2,...,k)$ is a subset of V_{cr}^-.

Now we prove a necessary and sufficient condition under which a vertex $v \in V_{cr}^0$.

Theorem 3.4. Let G be a semigraph and $v \in V(G)$ then $\alpha_s(G-v) = \alpha_s(G)$ if and only if there is a minimum strong vertex covering set S_1 of $G-v$ such that $N(v) \subset S_1$.

Proof: Suppose that $\alpha_s(G-v) = \alpha_s(G)$.

Let S be a $\alpha_s - set$ of G. If $v \in S$ then $\alpha_s(G-v) < \alpha_s(G)$, which is a contradiction. Thus, $v \notin S$ and therefore S is a strong vertex covering set of $G-v$. Hence, S is a $\alpha_s - set$ of $G-v$.

Let $S_1 = S$. Suppose $N(v) \not\subset S_1$ then there is a neighbour w of v such that $w \notin S_1$. Then $v \notin S; w \notin S$ which contradicts the fact that S is a strong vertex covering set of G. Thus, $N(v) \subset S_1$.

Conversely, suppose S_1 is an $\alpha_s - set$ of $G - v$ such that $N(v) \subset S_1$. We claim that S_1 is a strong vertex covering set of G. To prove this, suppose x and y are adjacent vertices in G. If $x \neq v$ and $y \neq v$, x and y are adjacent in $G - v$, then $x \in S_1$ or $y \in S_1$. If $x \neq v$ and $y \neq v$ and suppose x and y are adjacent in G but not in $G - v$, then every edge E which contains x and y also contains v. Therefore, x and y are neighbours of v and hence $x, y \in S_1$.

Suppose $x = v$ and $y \neq v$, then y is a neighbour of v, because x and y are adjacent in G. Hence, $y \in S_1$ as $N(v) \subset S_1$. Thus, $\alpha_s(G) \leq |S| \leq \alpha_s(G - v) \leq \alpha_s(G)$ which implies that $\alpha_s(G - v) = \alpha_s(G)$. This completes the proof of the theorem. ■

From the first part of the above theorem 3.5 it is clear that if $\alpha_s(G - v) = \alpha_s(G)$ then $N(v) \subset S_1$ for every $\alpha_s - set$ of G. Hence, we have the following corollary.

Corollary 3.5. If G is a semigraph $v \in V(G)$ and $\alpha_s(G - v) = \alpha_s(G)$ then

$$N(v) \subseteq \bigcap \{S : S \, is \, a \; \alpha_s - set \, of \; G\}.$$

Corollary 3.6. Let G be a semigraph then V_{cr}^0 is a strong independent subset of G.

Proof: Suppose u and $v \in V_{cr}^0$ with $u \neq v$. If u and v are adjacent then $u \in N(v)$ and hence by Corollary 3.5, $u \in S$, for every $\alpha_s - set$ S of G. Let S_0 be any $\alpha_s - set$ of G, then $u \in S_0$. Hence by Theorem 3.2, $u \in V_{cr}^-$, which is a contradiction as $v \in V_{cr}^0$. Therefore, u and v cannot be adjacent. Thus, V_{cr}^0 is a strong independent set. ■

Theorem 3.7. Let G be a semigraph and $v \in V(G)$. Then $\beta_s(G - v) < \beta_s(G)$ if and only if there is a maximum strong independent set T of $G - v$ such that $N(v) \cap T = \phi$.

Proof: Suppose $\beta_s(G - v) < \beta_s(G)$. Let T be a maximum independent set of G. If $v \in T$, then $T - \{v\}$ is a maximum independent set of $G - v$. Since v is not adjacent to any vertex in G, $N(v) \cap T = \phi$.

Suppose, $v \notin T$ for any maximum independent set T of G. Then for any such set T, T is an independent set in $G - v$, which implies that $\beta_s(G - v) \geq \beta_s(G)$, which is contradiction. Thus, it is impossible that $v \notin T$, for every maximum independent set T of G.

Conversely, suppose T is a maximum independent set of $G - v$ such that $N(v) \cap T = \phi$. We claim that T is an independent set in G also. Suppose $x, y \in T$ which are adjacent in G. Then $x \neq v$

and $y \neq v$ as $N(v) \cap T = \phi$. Then x and y are adjacent in $G - v$ which contradicts the maximum independence of T in $G - v$.

Let $T_1 = T \cup \{v\}$. Since, $N(v) \cap T = \phi$, T_1 is an independent set in G and hence, $\beta_s(G) \geq |T_1| > |T| = \beta_s(G - v)$. Thus, $\beta_s(G - v) < \beta_s(G)$. ■

Corollary 3.8. Let G be a semigraph and $v \in V(G)$ if $\beta_s(G - v) < \beta_s(G)$ then there is a maximum independent set S of G such that $v \in S$.

The converse of above corollary is not true in general.

Example 3.9. Consider the semigraph G whose vertex set $V(G) = \{1,2,3,...,7\}$ and edges are $(1,2,3), (1,6,4), (3,4), (3,6), (4,5)$ and (5,6). Note that 7 is an isolated vertex in G. We can observe that $S = \{1,5,7\}$ is a maximum independent set of G and $\beta_s(G) = 3$. Now consider the semigraph $(G - 1)$. The edges of this semigraph are (3,4),(3,6),(4,5) and (5,6). In this semigraph $T = \{1,5,7\}$ is a maximum independent set and hence $\beta_s(G - 1) = 3$. Thus, $\beta_s(G - v) = \beta_s(G)$ although $1 \in S$ which is maximum strong independent set of G.

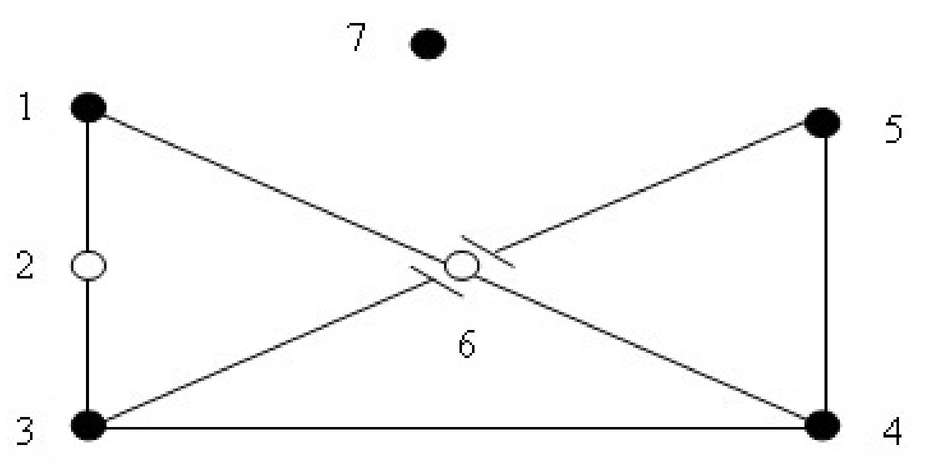

Figure 2

We may note that $\alpha_s(G - v) = \alpha_s(G) - 1$ is not always true if $\alpha_s(G - v) < \alpha_s(G)$.

Example 3.10. Consider the semigraph given in Figure 3. Here the set $S = \{0,1,2,3\}$ is an $\alpha_s - set$ of G. Hence, $\alpha_s(G) = 4$. In the semigraph $(G - 0)$ there are no edges and it vertex set is 1, 2, 3, 4, 5 and 6. Thus, $\alpha_s(G - 0) = 0$. Hence, $\alpha_s(G - 0) = \alpha_s(G) - 4$.

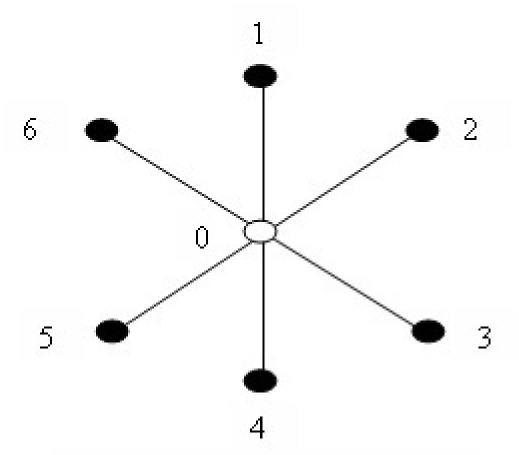

Figure 3

Theorem 3.11. Let G be a semigraph and $v \in V(G)$, then $\alpha_s(G-v) = \alpha_s(G) - k$ if and only if $\beta_s(G-v) = \beta_s(G) + k - 1$ for every integer $k \geq 0$ and $k \leq \alpha_s(G)$.

Proof: Let n be the number of the vertices of a semigraph G. Here $\alpha_s(G-v) + \beta_s(G-v) = n-1$. Suppose $\alpha_s(G-v) = \alpha_s(G) - k$. Then $\alpha_s(G) - k + \beta_s(G-v) = n-1$. Therefore, $\beta_s(G-v) = (n - \alpha_s(G)) + k - 1$. Hence, $\beta_s(G-v) = \beta_s(G) + k - 1$.

The converse can be proved in a similar manner. ■

Acknowledgement: The author of this paper would like to thank the reviewers for their valuable suggestions.

References

[1] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Domination in Graphs Advanced Topics*, Marcel Dekker, Inc, (1998).

[2] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamental of Domination in Graphs*, Marcel Dekker, Inc, (1998).

[3] E. Sampathkumar, *Semigraphs and their Applications*, Report on the DST (Department of Science and Technology) Project, 2000.

[4] Shyam S. Kamath and R. S. Bhat, *Domination in semigraphs*, Electronic Notes in Discrete Mathematics, Vol. 15(2003), 106-111.

[5] Shyam S. Kamath, Saroja R. Hebber and Ravishankar Bhat, *Domination in Semigraph (Part-I) Lecture Notes National Workshop on Semigraphs*, June 04-07, 2010.