International Journal of Mathematics and Soft Computing
Vol.4, No.1 (2014), 93 - 101.

ISSN Print : 2249 - 3328
ISSN Online: 2319 - 5215

Some new perspectives on odd sequential graphs

S K Vaidya
Saurashtra University,
Rajkot - 360 005, Gujarat, INDIA.
E-mail: samirkvaidya@yahoo.co.in

Lekha Bijukumar
Shanker Sinh Vaghela Bapu Institute of Technology,
Gandhinagar, Gujarat, INDIA.
E-mail: dbijuin@yahoo.co.in

Abstract

A graph $G = (V(G), E(G))$ with p vertices and q edges is said to be an *odd sequential graph* if there is an injection $f : V(G) \rightarrow \{0, 1, \ldots, q\}$ or if G is a tree then f is an injection $f : V(G) \rightarrow \{0, 1, \ldots, 2q - 1\}$ such that when each edge xy is assigned the label $f(x) + f(y)$, the resulting edge labels are $\{1, 3, ..., 2q - 1\}$. In this paper we investigate some new families of odd sequential graphs. We also introduce two new concepts namely bi-odd sequential graphs and global odd sequential graphs and some of their characteristics are discussed.

Keywords: Odd sequential labeling, step ladder graph, arbitrary super subdivision of a graph.
AMS Subject Classification(2010): 05C78.

1 Introduction

By a graph $G = (V(G), E(G))$ we mean a simple, connected and undirected graph in this paper. The terms not defined here are used in the sense of Harary[3]. In order to maintain the compactness, we give a summary of definitions.

Definition 1.1. Duplication of a vertex v_k of a graph G produces a new graph G_1 by adding a vertex $v_k^{'}$ in such a way that *N(v_k)=N($v_k^{'}$)*.

Definition 1.2. Let P_n be a path on n vertices denoted by $(1, 1), (1, 2), ..., (1, n)$ and with $n - 1$ edges denoted by $e_1, e_2, ..., e_{n-1}$ where e_i is the edge joining the vertices $(1, i)$ and $(1, i + 1)$. On each edge $e_i, i = 1, 2, ..., n - 1$ we erect a ladder with $n - (i - 1)$ steps including the edge e_i. The graph obtained is called *a step ladder graph* and is denoted by $S(T_n)$, where n denotes the number of vertices in the base.

Definition 1.3. [6] Let $G = (V(G), E(G))$ be a graph and G_1,G_2,...,G_n be n copies of graph G. Then the graph obtained by adding an edge between G_i and G_{i+1}, for $i = 1, 2, ..., n-1$ is called *a path union of* G.

Definition 1.4. Consider m copies of stars $K_{1,n}$ then $G = \left[\left\langle K_{1,n}^{(1)} : K_{1,n}^{(2)} : ... : K_{1,n}^{(m)}\right\rangle\right]$ is the graph obtained by joining the apex vertices of the stars $K_{1,n}^{(p-1)}$ and $K_{1,n}^{(p)}$ to a new vertex x_{p-1} where $1 \leq p \leq m$.

Definition 1.5. The *arbitrary super subdivisions* of a graph G produces a new graph by replacing each edge of G by an arbitrary complete bipartite graph K_{2,m_i} (where m_i is any positive integer) in such a way that the ends of each e_i are merged with the two vertices of 2-vertices part of K_{2,m_i} after removing each edge e_i from the graph G. It is denoted by $SS(G)$.

Definition 1.6. The *shadow graph* $D_2(G)$ of a connected graph G is obtained by taking two copies of G say G' and G'', then join each vertex u' in G' to the neighbours of the corresponding vertex u'' in G''.

Graph labeling was introduced by Rosa[5] is now one of the fascinating areas of research with applications ranging from social sciences to computer science and from neural network to bio-technology to mention a few. A systematic study on various applications of graph labeling is carried out by Bloom and Golomb[1]. The famous Ringel-Kotzig[4] graceful tree conjecture and many illustrious works on it brought a tide of different labeling techniques like harmonious labeling, odd graceful labeling, edge graceful labeling and the like. For detailed survey on graph labeling and related results we refer to Gallian[2]. The concept of odd sequential labeling was introduced by Singh and Varkey[7] which is defined as follows.

Definition 1.7. A graph $G = (V(G), E(G))$ with p vertices and q edges is said to be an *odd sequential graph* if there is an injection $f : V(G) \to \{0, 1, \dots, q\}$ or if G is a tree then f is an injection $f : V(G) \to \{0, 1, \dots, 2q-1\}$ such that when each edge xy is assigned the label $f(x)+f(y)$, the resulting edge labels are $\{1, 3, ..., 2q-1\}$.

The graph that admits an odd sequential labeling is known as an *odd sequential graph*. In [7] it has been also proved that the graphs such as combs, grids, stars and rooted trees of level 2 are odd sequential while odd cycles are not.
In this paper we investigate some results on odd sequential labeling and also introduce two new concepts namely bi-odd sequential graphs and global odd sequential graphs.

2 Results on odd sequential labeling

Theorem 2.1. The path P_n admits odd sequential labeling.

Proof: Let v_1,v_2,..., v_n be the vertices of P_n. Define $f : V(P_n) \to \{0, 1, \dots, 2q-1\}$ as $f(v_i) = i-1$; $1 \le i \le n$. Then f is an odd sequential labeling for P_n. That is, P_n is an odd sequential graph. ■

Theorem 2.2. The cycle C_n admits odd sequential labeling for $n \equiv 0(mod4)$.

Proof: Let v_1,v_2,..., v_n be the vertices of the cycle C_n. Define $f : V(C_n) \to \{0, 1, \dots, q\}$ as follows.

$f(v_1) = 0$,
$f(v_2) = 1$,
$f(v_i) = n - i + 3$; for $3 \le i \le \frac{n+4}{2}$.
For $\frac{n+6}{2} \le i \le n$
$f(v_i) = n - i + 1$; if i is odd.
$\quad = n - i + 3$; if i is even.

The above defined function f is an odd sequential labeling for C_n for $n \equiv 0(mod4)$. That is, C_n is an odd sequential graph for $n \equiv 0(mod4)$. ∎

Illustration 2.3. The following figure shows an odd sequential labeling of cycle C_{12}.

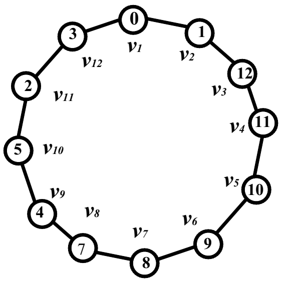

Figure 1: *An odd sequential labeling of* C_{12}.

Theorem 2.4. The crown $C_n \, [\odot] \, K_1$ is an odd sequential graph for even n.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle C_n and $u_1, u_2, \ldots, u_n$ be the newly added pendant vertices where n is even. Let G be $C_n \, [\odot] \, K_1$ with $p = 2n$ and $q = 2n$. To define $f : V(G) \to \{0, 1, \ldots, q\}$ we consider the following two cases.

Case 1: $n \equiv 0(mod4)$.

For $1 \leq i \leq \frac{n}{2} + 1$

$f(v_i) = 2i - 1$; if i is odd.

$= 2i - 2$; if i is even.

For $\frac{n}{2} + 2 \leq i \leq n$

$f(v_i) = 2i$; if i is even.

$= 2i - 1$; if i is odd.

For $1 \leq i \leq \frac{n}{2}$

$f(u_i) = 2i - 2$; if i is odd.

$= 2i - 1$; if i is even.

For $\frac{n}{2} + 1 \leq i \leq n$

$f(u_i) = 2i$; if i is odd.

$= 2i - 1$; if i is even.

Case 2: $n \equiv 2(mod4)$.

For $1 \leq i \leq \frac{n}{2}$

$f(v_i) = 2i - 2$; if i is odd.

$= 2i - 1$; if i is even.

$f(v_i) = 2i + 1$; for $i = \frac{n}{2} + 1$

For $\frac{n+4}{2} \leq i \leq n$

$f(v_i) = 2i$; if i is odd.

$= 2i - 1$; if i is even.

For $1 \leq i \leq \frac{n}{2} + 1$

$f(u_i) = 2i - 1$; if i is odd.

$= 2i - 2$; if i is even.
$f(u_i) = 2i - 3$; for $i = \frac{n}{2} + 2$
For $\frac{n}{2} + 3 \leq i \leq n$
$f(u_i) = 2i$; if i is even.
$= 2i - 1$; if i is odd.

In both the cases described above the graph under consideration admits an odd sequential labeling. That is, the crown $C_n\,[\odot]\,K_1$ is an odd sequential graph when n is even. ∎

Illustration 2.5. An odd sequential labeling for $C_{12}\,[\odot]\,K_1$ is given in *Figure 2.*

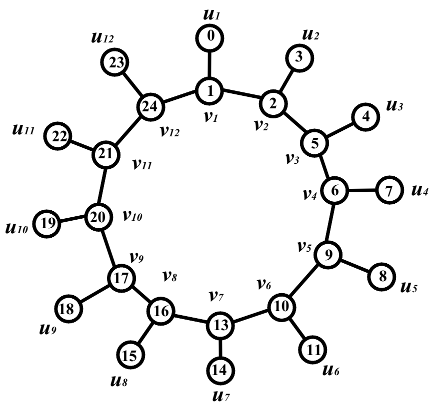

Figure 2: *An odd sequential labeling for* $C_{12}\,[\odot]\,K_1$.

Theorem 2.6. The graph obtained by the duplication of an arbitrary vertex in an even cycle C_n admits odd sequential labeling.

Proof: Let v_1,v_2,..., v_n be the vertices of the cycle C_n where n is even. Without loss of generality assume that the vertex v_1 is getting duplicated by a new vertex $v_1^{'}$. Let G be the resultant graph with $p = n + 1$ and $q = n + 2$. To define $f : V(G) \rightarrow \{0, 1, \ldots, q\}$ we consider the following two cases.

Case 1: $n \equiv 0(mod 4)$.
$f(v_1) = 3$,
$f(v_1') = 1$ $f(v_i) = n - i + 4$; for $2 \leq i \leq \frac{n}{2} + 1$.
For $\frac{n}{2} + 2 \leq i \leq n$:
$f(v_i) = n - i$; if i is even.
$= n - i + 4$; if i is odd.

Case 2: $n \equiv 2(mod 4)$.
$f(v_1) = 0$,
$f(v_1') = 2$,
$f(v_2) = 1$,
$f(v_i) = n - i + 5$; for $3 \leq i \leq \frac{n}{2} + 3$.
For $\frac{n}{2} + 4 \leq i \leq n$:
$f(v_i) = n - i + 3$; if i is odd.
$= n - i + 5$; if i is even.

In both the cases f is an odd sequential labeling. That is, the graph obtained by the duplication of an arbitrary vertex in an even cycle C_n admits odd sequential labeling. ∎

Illustration 2.7. An odd sequential labeling for the graph obtained by the duplication of an arbitrary vertex in an even cycle C_{12} is given in *Figure 3.*

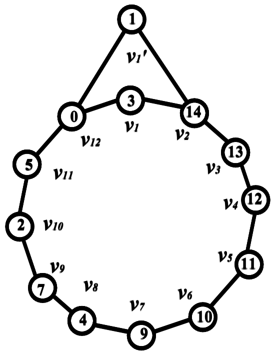

Figure 3: *An odd sequential labeling for the graph obtained by the duplication of an arbitrary vertex in an even cycle C_{12}.*

Theorem 2.8. The step ladder graph is an odd sequential graph.

Proof: Let P_n be a path on n vertices denoted by $(1,1),(1,2),...,(1,n)$ and with $n-1$ edges denoted by $e_1, e_2, ..., e_{n-1}$ where e_i is the edge joining the vertices $(1,i)$ and $(1,i+1)$. The step ladder graph $S(T_n)$ has $\frac{n^2+3n-2}{2}$ vertices denoted by $(1,1),(1,2),...,(1,n),(2,1),(2,2),...,(2,n)$, $(3,1),(3,2),...,(3,n-1),(4,1),(4,2),...,(4,n-2),...,(n,1),(n,2)$ and n^2+n+2 edges. In any ordered pair (i,j), i denotes the row (counted from bottom to top) and j denotes the column (from left to right) in which the vertex occurs. Define $f : V(S(T_n)) \rightarrow \{0,1,\ldots,q\}$ as follows.
$f(i,j) = n^2 - 2n(i-1) + i(i-2) + j - 1$; for $1 \leq i, j \leq n$, with $i + j \leq n + 2$. Then f provides an odd sequential labeling for $S(T_n)$. That is, $S(T_n)$ is an odd sequential graph. ∎

Illustration 2.9. The *Figure 4* shows an odd sequential labeling for $S(T_6)$.

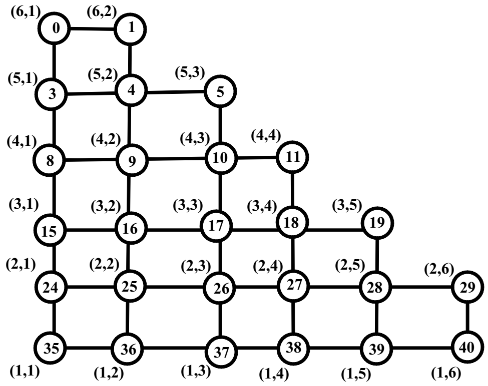

Figure 4: *An odd sequential labeling of $S(T_6)$.*

Theorem 2.10. The path union of stars is an odd sequential graph.

Proof: Let v_1,v_2,..., v_n be the vertices of P_n. Consider n copies of star $K_{1,m}$ and let $\{v_{ij}, 1 \leq i \leq n, 1 \leq j \leq m\}$ be their corresponding pendant vertices. Let G be the path union graph of n copies of star $K_{1,m}$ with $p = n(m+1)$ and $q = n(m+1) - 1$. $f : V(G) \to \{0, 1, \ldots, 2q-1\}$ as follows.
For $1 \leq i \leq n$:
$f(v_i) = i(m+1) - 1$; if i is even.
$= (i-1)(m+1)$; if i is odd.
For $1 \leq i \leq n, 1 \leq j \leq m$:
$f(v_{ij}) = (i-1)(m+1) + 2j - 1$; if i is odd.
$= i(m+1) - 2(m-j+1)$; if i is even.
The above defined function f provides an odd sequential labeling for the path union of stars. That is, the path union of stars is an odd sequential graph. ■

Illustration 2.11. The following figure shows an odd sequential labeling for the path union of four copies of star $K_{1,4}$.

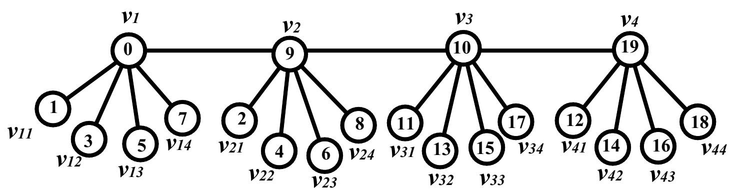

Figure 5: *An odd sequential labeling of the path union of four copies of star* $K_{1,4}$.

Theorem 2.12. The graph $G = \left[\left\langle K_{1,n}^{(1)} : K_{1,n}^{(2)} : ... : K_{1,n}^{(m)} \right\rangle\right]$ admits odd sequential labeling.

Proof: Consider m copies of star $K_{1,n}$ and let G be the graph $G = \left[\left\langle K_{1,n}^{(1)} : K_{1,n}^{(2)} : ... : K_{1,n}^{(m)} \right\rangle\right]$. Let $V(G) = \{v_1; i = 1,2,...m, v_{ij}; i = 1,2,...m, j = 1,2,...n, x_i; i = 1,2,...m-1\}$ and $E(G) = \{v_i v_{ij}; i = 1,2,...m, j = 1,2,...n, v_i x_i; i = 1,2,...m-1, x_i v_{i+1}; i = 1,2,...m-1\}$ so that $p = m(n+2) - 1$ and $q = m(n+2) - 2$. Define $f : V(G) \to \{0, 1, \ldots, 2q-1\}$ as follows.
$f(v_i) = 2i - 2; 1 \leq i \leq m,$
$f(v_{1j}) = 2j - 1; 1 \leq j \leq n,$
$f(v_{ij}) = 2i(n+1) - 2(n-j) - 3, 2 \leq i \leq m; 1 \leq j \leq n$
$f(x_i) = 2i(n+1) - 1; 1 \leq i \leq m-1.$
The above defined function f provides an odd sequential labeling for $\left[\left\langle K_{1,n}^{(1)} : K_{1,n}^{(2)} : ... : K_{1,n}^{(m)} \right\rangle\right]$. That is, the graph $\left[\left\langle K_{1,n}^{(1)} : K_{1,n}^{(2)} : ... : K_{1,n}^{(m)} \right\rangle\right]$ is an odd sequential graph. ■

Illustration 2.13. *Figure 6* shows an odd sequential labeling for the graph $\left[\left\langle K_{1,4}^{(1)} : K_{1,4}^{(2)} : K_{1,4}^{(3)} \right\rangle\right]$.

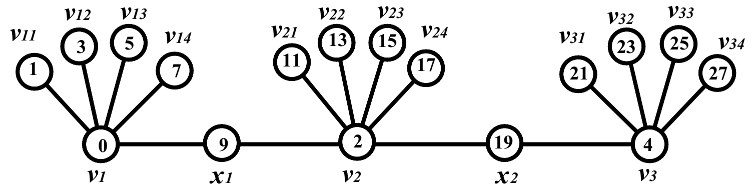

Figure 6: *An odd sequential labeling of* $\left[\left\langle K_{1,4}^{(1)} : K_{1,4}^{(2)} : K_{1,4}^{(3)}\right\rangle\right]$.

Theorem 2.14. The graph $SS(P_n)$ admits odd sequential labeling.

Proof: Let P_n be the path containing n vertices v_1,v_2,..., v_n and $n-1$ edges. Let e_i denotes the edge v_iv_{i+1} in P_n. For $1 \leq i \leq n-1$ each edge e_i of path P_n is replaced by a complete bipartite graph K_{2,m_i} where m_i is any positive integer. Let u_{ij} be the vertices of the m_i vertices part of K_{2,m_i} where $1 \leq i \leq n-1$,$1 \leq j \leq m_i$. Define $f : V(SS(P_n)) \to \{0, 1, \ldots, q\}$ as follows.
$f(v_1) = 0$,
$f(v_i) = 2\sum_{k=1}^{i-1} m_k$; $2 \leq i \leq n$,
$f(u_{1j}) = 2j - 1$; $1 \leq j \leq m_1$,
$f(u_{ij}) = 2\sum_{k=1}^{i-1} m_k + 2j - 1$; $2 \leq i \leq n$,$1 \leq j \leq m_i$.
In view of the above defined labeling pattern f is an odd sequential labeling for $SS(P_n)$. That is, $SS(P_n)$ is an odd sequential graph. ∎

Illustration 2.15. An odd sequential labeling for $SS(P_5)$ is shown in Figure 7.

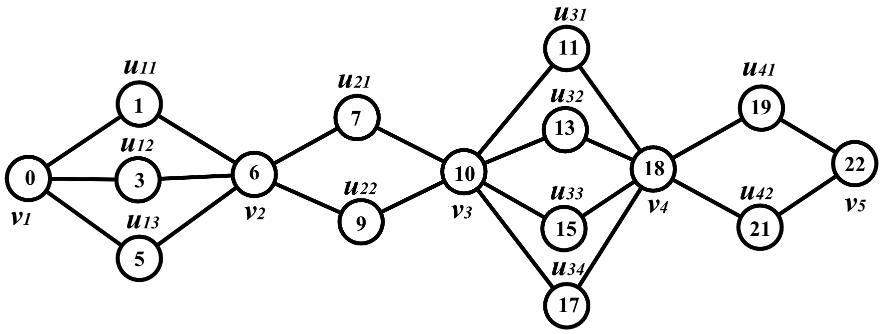

Figure 7: An odd sequential labeling of $SS(P_5)$.

Theorem 2.16. The graph $D_2(K_{1,n})$ is an odd sequential graph.

Proof: Let $K'_{1,n}$ and $K''_{1,n}$ be two copies of star $K_{1,n}$. Let v',v'_1,v'_2,..., v'_n be the vertices of $K'_{1,n}$ and v'',v''_1,v''_2,..., v''_n be the vertices of $K''_{1,n}$ where v',v'' be the corresponding apex vertices. Define $f : V(D_2(K_{1,n})) \to \{0, 1, \ldots, q\}$ as follows.
$f(v'') = 0$,
$f(v''_i) = 2i - 1$; $1 \leq i \leq n$,
$f(v') = 4n$ $f(v'_i) = 4n - 2i + 1$; $1 \leq i \leq n$.

The above defined function f is an odd sequential labeling for $D_2(K_{1,n})$. That is, $D_2(K_{1,n})$ is an odd sequential graph. ■

Illustration 2.17. An odd sequential labeling of $D_2(K_{1,4})$ is shown in *Figure 8*.

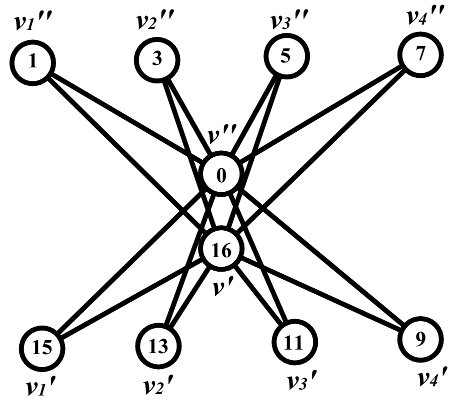

Figure 8: *An odd sequential labeling for* $D_2(K_{1,4})$.

3 Bi-odd sequential and global odd sequential graphs

Definition 3.1. A graph G is said to be a bi-odd sequential graph if both G and its line graph $L(G)$ admit odd sequential labeling.

Theorem 3.2. The path P_n for all n and the cycle C_n for $n \equiv 0(mod4)$ are bi-odd sequential graphs.

Proof: $L(P_n)$ and $L(C_n)$ are isomorphic to P_n and C_n respectively. Hence the theorem follows in view of Theorem 2.1 and Theorem 2.2. ■

Theorem 3.3. A tree is bi-odd sequential if and only if it is a path.

Proof: Let G be a tree which is bi-odd sequential. Then G and its line graph $L(G)$ admit odd sequential labeling. But all the trees with $p \geq 4$ except P_n contain atleast one $K_{1,3}$ and $L(K_{1,3)}$ is C_3 which is a forbidden sub graph for a graph to be odd sequential. Therefore, a tree is bi-odd sequential if it is path P_n.
Conversely let the tree be path P_n. In view of Theorem 3.2, P_n is bi-odd sequential. ■

Definition 3.4. A graph G is said to be a global odd sequential graph if both G and its complement G^c admit odd sequential labeling.

Theorem 3.5. P_4 is the only global odd sequential graph.

Proof: Let G be an odd sequential graph with p vertices. Consider the following two cases.
Case 1: $p < 6$.
Then G^c is either a totally disconnected graph, $2K_2$ or a graph contains C_3 except for P_4 and these three graphs fail to be odd sequential. Since P_4 is a self complementary graph it is an odd sequential graph.

Case 2: $p \geq 6$.
According to Ramsey theory, if G is a graph with $p \geq 6$, then either G or G^c contains a triangle. Since all the odd cycles are forbidden sub graphs for a graph to be odd sequential, any graph with $p \geq 6$ can not be a global odd sequential graph.
From the above two cases, it is clear that P_4 is the only global odd sequential graph. ■

4 Concluding Remarks

This paper presents some new families of odd sequential graphs and the new concepts namely bi-odd sequential graphs and global odd sequential graphs. We also establish a characterisation for bi-odd sequential graph. We also show that P_4 is the only global odd sequential graph.

Acknowledgement: The authors are highly thankful to the anonymous referee for the constructive comments and suggestions.

References

[1] G.S. Bloom and S.W. Golomb, *Applications of numbered undirected graphs*, Proceeding of IEEE, 65(4)(1977), 562-570.

[2] J A Gallian, *A dynamic survey of graph labeling*, The Electronics Journal of Combinatorics, 18(2011) $\sharp DS6$.

[3] F Harary, *Graph theory*, Addison Wesley, Reading, Massachusetts, 1972.

[4] G Ringel, *Problem 25, Theory of graphs and its applications*, Proc. Symposium Smolenice 1963, Prague (1964), 162.

[5] A Rosa, *On certain valuation of the vertices of a graph*, Theory of graphs, International Symposium, Rome, July (1966), Gorden and Breach, New York and Dunod Paris(1967), 349-355.

[6] S C Shee and Y S Ho, *The cordiality of the path union of n copies of a graph*, Discrete Mathematics, 151(1996), 221-229.

[7] G S Singh and T K M Varkey, *On odd sequential and bi sequential graphs*, Preprint.