The All-Ones Problem for Binomial Trees, Butterfly and Benes Networks

Paul Manuel
Department of Computer Science,
Kuwait University, KUWAIT.

Indra Rajasingh, Bharathi Rajan
Department of Mathematics,
Loyola College, Chennai 600034, INDIA.

R. Prabha
Department of Mathematics,
Ethiraj College for women, Chennai-600 008, INDIA.
E-mail: prabha75@gmail.com

Abstract
The all-ones problem is an NP-complete problem introduced by Sutner [11], with wide applications in linear cellular automata. In this paper, we solve the all-ones problem for some of the widely studied architectures like binomial trees, butterfly, and benes networks.

Keywords: all-ones problem, dominating set, binomial trees, butterfly networks.

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1 Introduction

The All-Ones Problem was introduced by Sutner [11] where he also discussed wide applications of the all-ones problem in linear cellular automata. The problem has been studied under various names by various authors in different papers. The all-ones problem can be stated as follows: Suppose each square of an $n \times n$ chessboard is equipped with an indicator light and a button. If the button of a square is pressed, the light of that square will change from off to on, and vice versa; the same happens to the lights of all the edge-adjacent squares. Initially all lights are off. Now, we consider the following questions: is it possible to press a sequence of buttons in such a way that in the end all lights are on? This is referred to as the all-ones problem. If there is such a solution, how can we find it? And finally, how can we find a solution that presses as few buttons as possible? This is referred to as the minimum all-ones problem.

The above questions can be posed for any arbitrary graph. In this chapter, we consider connected, simple, undirected graphs only. One can deal with disconnected graphs, component by component. For all notations and terminology used in this chapter, we refer to [2]. An equivalent version of the all-ones problem was proposed by Peled [8], where it was called the Lamp Lighting problem. The rule of the all-ones problem is called $\sigma^+$ rule on graphs, which means that a button lights not only its neighbours but also its own light. If a button lights only its neighbours but not its own light, this rule on graphs is called $\sigma$ rule.

There have been many publications on the All-Ones Problem [1, 6, 12]. Using linear algebra, Sutner [13] proved that it is always possible to light every lamp in any graph by $\sigma^+$ rule. A graph-theoretic proof was given by Eriksson et al. [7]. In [9] Sutner proved that the Minimum all-ones problem is
NP-complete in general. Li et al. [3] have proved that the problem is NP-complete even when restricted to bipartite graphs. Li et al. [5] have given a linear time algorithm for finding optimal solutions for trees.

2 Notations

Let $G = (V, E)$ be a simple, connected graph. Let $|V| = n$ and $|E| = m$. The open neighbourhood of a vertex $v$ is $\{u \in V : u \text{ is adjacent to } v\}$ and is denoted by $N(v)$. The closed neighbourhood of a vertex $v$ is $\{v\} \cup N(v)$ and is denoted by $N[v]$. The degree of a vertex $v$ in $G$ is the number of neighbours of $v$ in $G$. The all-ones problem is equivalent to the following dominating set problem [4] from a graph-theoretic point of view.

A set $S$ of vertices is independent if no two vertices in $S$ are adjacent in $G$. A clique is a graph where all the vertices are mutually adjacent. A maximal clique (independent set) of $G$ is one in which we cannot add a vertex and still have a clique(independent set) in $G$. For any $X \subseteq V$ if $G[V \setminus X]$ has more than one connected component, we say that $X$ is a separator of $G$. A set $S$ of vertices in a graph $G$ is a dominating set if every vertex, not in $S$, is adjacent to at least one vertex in $S$.

**Definition 2.1.** Given a graph $G = (V, E)$, where $V$ and $E$ denote the vertex-set and the edge-set of $G$ respectively, the all-ones problem is to find a subset $S \subseteq V$ with the property that every vertex in $S$ has an even number of neighbours in $S$, while every vertex in $V \setminus S$ has an odd number of neighbours in $S$. Since this implies that every vertex is dominated by an odd number of vertices in $S$ (including the vertex itself if it belongs to $S$), the solution set $S$ is called an odd dominating set or a $OD$-set of $G$. If we are able to find such a set with minimum cardinality, then such an $S$ is said to be a solution to the minimum all-ones problem and is called a minimum all-ones dominating set or typically a $MOD$-set of $G$.

3 Binomial Trees

In this section we solve the minimum all-ones problem for the binomial trees.

A binomial tree of height 0 is a single vertex. For all $h > 0$, a binomial tree of height $h$ is a tree formed by joining the roots of two binomial trees of height $h − 1$ with a new edge and designating one of these roots to be the root of the new tree. A binomial tree of height $n$ has $2^n$ vertices.

![Figure 1: MOD-set of $B_2$ and $B_3$.](image)

**Lemma 3.1.** Let $B_n$ ($n > 2$) be a binomial tree which contains two binomial trees of height $n − 1$ with roots $u$ and $v$ respectively. If $S$ is a solution to the minimum all-ones problem (MOD-set) of $B_n$, then both $u$ and $v$ cannot lie in $S$.

**Proof.** Suppose $u$ lies in $S$. By definition of binomial tree, $u$ lies in some $B_2$. Consider the vertex of degree 2 in that $B_2$, adjacent to $u$ and let us call it $w$. Let the pendant vertex adjacent to $w$ be $x$ (refer
Figure 1). Since \( x \) is of degree 1, either \( w \) or \( x \) should lie in \( S \). Since \( u \) lies in \( S \), in either case, \( w \) will be evenly dominated by \( S \) which is a contradiction. By a similar argument we can prove that \( v \) also cannot lie in \( S \).

**Lemma 3.2.** Let \( B_n \) be a binomial tree which contains two binomial trees of height \( n - 1 \) say \( B_{n-1}' \) and \( B_{n-1}'' \) with \( MOD \)-sets \( S_1 \) and \( S_2 \) respectively. Then the \( MOD \)-set for \( B_n \) is \( S = S_1 \cup S_2 \).

**Proof.** Let \( W \) be any other odd-dominating set of \( B_n \). Suppose \( W_1 = W \cap V(B_{n-1}') \) and \( W_2 = W \cap V(B_{n-1}'') \). Then by Lemma 3.1, the roots of \( B_{n-1}' \) and \( B_{n-1}'' \) cannot lie in \( W \). Hence \( W_1 \) and \( W_2 \) will necessarily be odd-dominating sets of \( B_{n-1}' \) and \( B_{n-1}'' \) respectively. Since \( S_1 \) and \( S_2 \) are \( MOD \)-sets of \( B_{n-1}' \) and \( B_{n-1}'' \) respectively, \( |S_1| \leq |W_1| \) and \( |S_2| \leq |W_2| \). Hence \( |S_1 \cup S_2| \leq |W_1 \cup W_2| \) proving that \( |S| \leq |W| \).

**Theorem 3.3.** The solution to the minimum all-ones problem for a binomial tree \( B_n \) is the set of all its pendant vertices.

**Proof.** The proof is by induction on \( n \). Consider the case \( n = 2 \). \( B_2 \) is isomorphic to \( P_4 \). Hence the \( MOD \)-set of \( B_2 \) are its two terminal vertices. Using Lemma 3.2 repeatedly by induction, one can easily find the required \( MOD \)-set of \( B_n \) for all \( n \).

### 4 Butterfly networks

In this section, we solve the minimum all-ones problem for butterfly architectures.

The vertices of the \( n \)-dimensional butterfly network \((BF_n)\) are the pairs \((r, x)\), where \( r \) is a non-negative integer \( 0 \leq r \leq n \) called the rank, and \( x = x_1x_2...x_n \) is a binary string of length \( n \). A vertex \((r, x), 0 \leq r \leq n - 1\), is joined to the vertices \((r + 1, x)\) and \((r + 1, x_1x_2...x_rx_{r+1}x_{r+2}...x_n)\).

The edges of the form \(((r, x), (r + 1, x))\) are called straight edges, edges of the form \(((r, x), (r + 1, x_1x_2...x_rx_{r+1}x_{r+2}...x_n))\) are called cross edges and the edges connecting vertices on ranks \( i \) and \( i + 1 \) are called level \( i \) edges. The Butterfly network \((BF_n)\) has \((n + 1)2^n\) vertices.

Consider \( BF_{k+1} \). It contains two \( k \)-dimensional butterflies say \( BF_k' \) and \( BF_k'' \). Let \( B = \{ v \in V(BF_{k+1}) | v \notin V(BF_k') \text{ and } v \notin V(BF_k'') \} \). The vertices in \( B \) can be partitioned into ordered pairs \((x, y)\) lying on the same 4-cycle such that \( d(x) = d(y) = 2 \) and \( d(x, y) = 2 \). We call such an ordered pair of vertices as 2-distance, 2-degree vertices or a “2DD-pair” in short. Also if \((x, y)\) is a 2DD-pair, then \( N(x) = N(y) \). See Figure 2.

![Figure 2: A 2DD-pair of BF2 and BF3.](image-url)
Lemma 4.1. If \((x, y)\) is a 2DD-pair of \(BF_{k+1}\), then either \(x\) and \(y\) are in \(S\) or \(x\) and \(y\) are not in \(S\).

**Proof.** Let \(N(x) = N(y) = \{a, b\}\) such that \(a \in V(BF_k')\) and \(b \in V(BF_k'')\). Suppose only one vertex of the 2DD-pair \((x, y)\) lies in \(S\). Without loss of generality say \(x \in S\) and \(y \notin S\). Since \(x \in S\), either both \(a\) and \(b\) are in \(S\) or both \(a\) and \(b\) are not in \(S\) as \(x\) has to be odd-dominated by \(S\).

**Case 1:** \(a \in S\) and \(b \in S\).

Then \(y\) is evenly-dominated by \(S\) which contradicts our hypothesis that \(S\) is an odd-dominating set.

**Case 2:** \(a\) and \(b\) are not in \(S\).

In this case, \(y\) is not dominated by \(S\) at all, contradicting our hypothesis that \(S\) is a solution set. ■

Lemma 4.2. Let \((x, y)\) be a 2DD-pair of \(BF_{k+1}\) and \(N(x) = N(y) = \{a, b\}\) such that \(a \in V(BF_k')\) and \(b \in V(BF_k'')\). If \(x\) and \(y\) are not in \(S\), then either \(a \in S\) or \(b \in S\).

**Proof.** \((x, y)\) is a 2DD-pair of \(BF_{k+1}\). Suppose if both \(a\) and \(b\) are in \(S\), then \(x\) and \(y\) will be dominated by an even number of vertices in \(S\). If both \(a\) and \(b\) are not in \(S\), then, both \(x\) and \(y\) are not dominated by \(S\) at all. ■

**Theorem 4.3.** \(V(BF_r)\) is the only solution to the all-ones problem of the \(r\)-dimensional butterfly \(BF_r\).

**Proof.** The proof is by induction on the dimension \(r\) of \(BF_r\). For \(r = 1\), the proof is trivial. Assume that the result is true for \(r = k\). Consider \(BF_{k+1}\). It contains two \(k\)-dimensional butterflies say \(BF'_k\) and \(BF''_k\). Let \(S, S'\), and \(S''\), be the solution sets to the minimum all-ones problem for \(BF_{k+1}\), \(BF'_k\) and \(BF''_k\) respectively. By induction hypothesis, \(S' = V(BF'_k)\) and \(S'' = V(BF''_k)\). Now let \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\) where \(n = 2k\), be the 2DD-pairs of \(BF_{k+1}\). By Lemma 4.1, if \((x_i, y_i)\) is a 2DD-pair of \(BF_{k+1}\) then \(x_i\) and \(y_i\) are in \(S\) or \(x_i\) and \(y_i\) are not in \(S\). Suppose there exists a 2DD-pair \((x_i, y_i)\) such that \(x_i\) and \(y_i\) are not in \(S\). Then by Lemma 4.2, either \(a_i \in S\) or \(b_i \in S\). Without loss of generality let \(a_i \in S\) and \(b_i \notin S\). Now suppose \(b_i \notin V(BF''_k)\). Then, \(x_i\) and \(y_i\) are not in \(S\) implies that \(b_i\) is being dominated by vertices of \(BF''_k\) only. This in turn implies that \(S'' \subseteq V(BF''_k)\) which is a contradiction to our induction hypothesis that \(S'' = V(BF''_k)\). ■

5 Benes networks

The \(r\)-dimensional Benes network consists of back-to-back butterflies, denoted by \(B(r)\). The \(B(r)\) has \(2r + 1\) levels, each with \(2r\) vertices. The first and last levels in the \(B(r)\) form two butterflies \(BF_r\) respectively, while the middle level is shared by these butterfly networks. The \(r\)-dimensional Benes network has \((n + 1)2^{2n+1}\) vertices and \(n2^{n+2}\) edges.

The level zero to level \(r\) vertices in the network form an \(r\)-dimensional butterfly. The middle level of the Benes network is shared by these butterflies An \(r\)-dimensional Benes is denoted by \(B(r)\). Figure 3 shows a \(B(2)\) network. A Benes network is bipartite. No two 4-cycles of \(B(r)\) have a common edge. The edge set of \(B(r)\) is disjoint union of 4-cycles.
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Figure 3: Diamond representation of $B(2)$.

**Theorem 5.1.** $V(B(r))$ is the only solution to the all-ones problem of the $r$-dimensional Benes $B(r)$.

**Proof.** The benes network consists of back-to-back butterflies. Hence similar to the butterfly networks, we identify all the $2DD$-pairs of the benes networks. Tracing the same steps as in Theorem 4.3, we obtain the desired result.

**References**


