Clique transversal sets

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Abstract

Let \( G = (V, E) \) be a simple graph. A clique of \( G \) is a maximal complete subgraph of \( G \). A subset \( D \) of \( V \) which meets every clique of \( G \) is called a clique transversal set of \( G \). Since \( V(G) \) is a clique transversal set of \( G \), the existence of a clique transversal set of \( G \) is guaranteed in any graph. The minimum cardinality of a clique transversal set of \( G \) is called the clique transversal number of \( G \) and is denoted by \( \gamma_{ct}(G) \). A study on this concept is initiated in this paper.

Keywords: Clique transversal sets, clique transversal number.

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1 Introduction

Transversals with specific properties in graphs have been studied widely. Chromatic transversals, clique transversals are some of the transversal sets which have received the attention of many researchers. In this paper, we study the clique transversal sets.

Definition 1.1. A set \( S \subseteq V \) of a graph \( G \) is a clique transversal set of \( G \) if \( S \) intersects every clique of \( G \).

Definition 1.2. The minimum cardinality of a clique transversal set of \( G \) is called clique transversal number of \( G \) and is denoted by \( \gamma_{ct}(G) \).

Remark 1.3. Any clique transversal set of \( G \) is a dominating set of \( G \). For: Let \( S \) be a clique transversal set of \( G \). Let \( u \in V(G)-S \). Then \( u \) is contained in a clique say \( C \) of \( G \). Since \( S \) is a clique transversal set of \( G \), \( S \cap C \neq \emptyset \). Let \( u \in S \cap C \). Then \( v \) dominates \( u \). Therefore, \( S \) is a dominating set of \( G \).

Remark 1.4. A Clique transversal set of \( G \) need not be a total dominating set of \( G \). To illustrate this consider the following graph in Figure 1. \( S = \{v_3, v_6, v_8\} \) is a clique transversal set of \( G \). \( S \) is a dominating set of \( G \) but \( S \) is not a total dominating set of \( G \).
Result 1.5. Clique transversal numbers of some standard graphs are given in the following.

(i). $\gamma_{clt}(K_n) = 1$.

(ii). $\gamma_{clt}(K_{1,n}) = 1$.

(iii). $\gamma_{clt}(P_n) = \alpha_0(P_n) = \left[ n/2 \right]$.

(iv). $\gamma_{clt}(C_n) = \alpha_0(C_n) = \lceil n/2 \rceil$, $n \geq 4$.

(v). $\gamma_{clt}(G) = \alpha_0(G)$ if and only if $G$ is triangle free.

(vi). $\gamma_{clt}(W_n) = 1$.

(vii). $\gamma_{clt}(D_{r,s}) = 1$.

Remark 1.6.

(i) Suppose $G$ is a disconnected graph with components $G_1$, $G_2$, ..., $G_k$. Then $\gamma_{clt}(G) = \sum_{i=1}^{k} \gamma_{clt}(G_i)$.

(ii) If the graph $G$ has a full degree vertex, then $\gamma_{clt}(G) = \gamma(G) = 1$.

(iii) If the graph $G$ has $t$-disjoint maximum cliques each of order $k$, then $\gamma_{clt}(G) \leq t + \gamma(G-H)$, where $H$ is the subgraph of $G$ formed by the union of the $t$-disjoint maximum cliques and the $t$-vertices one from each of the disjoint maximum cliques and so chosen that these vertices dominate maximum number of vertices in $V(G-H)$. The equality is reached if

(a) $G - H = \emptyset$. That is, $G$ is the disjoint union of complete graph of some order in $G$.

(b) $V(G-H)$ is dominated by the $t$-vertices that are selected from the $t$-disjoint components.

2 Bounds for $\gamma_{clt}(G)$

Theorem 2.1. (i) $1 \leq \gamma_{clt}(G) \leq n$. $\gamma_{clt}(G) = n$ if and only if $G = \overline{K}_n$ and $\gamma_{clt}(G) = 1$ if and only if $G$ has a full degree vertex.

(ii) $\gamma_{clt}(G) = n - 1$ if and only if $G = \overline{K}_{n-2} \cup K_2$.

Proof: (i) Obvious.
(ii) If $G = K_{n-2} \cup K_2$, then $\gamma(G) = n-1$. Also any minimum dominating set of $G$ intersects every clique of $G$. Therefore, $\gamma_{clt}(G) = n-1$. Conversely, let $\gamma_{clt}(G) = n-1$. Suppose $G$ has a clique $S$ of cardinality greater than or equal to 3, then $\gamma_{clt}(G) \leq n-2$, a contradiction. Therefore, any clique of $G$ has atmost two vertices. Suppose there are two disjoint cliques $S_1$ and $S_2$ of cardinality 2, then also, $\gamma_{clt}(G) \leq n-2$, a contradiction. Therefore, $G$ has atmost one clique of order 2. Suppose $G$ has an induced $P_3$, then also $\gamma_{clt}(G) \leq n-2$, a contradiction. Therefore, $G$ has exactly one $K_2$ and all other vertices are independent. That is, $G = K_{n-2} \cup K_2$. 

**Theorem 2.2.** $\gamma_{clt}(G) = 2$ if and only if there exists two vertices $u, v \in V(G)$ such that $G$ has at least three cliques and any two cliques of $G$ have either $u$ as a common point or $v$ as a common point and there are two cliques which have $u$ as a common point and not $v$, another two cliques which have $v$ as a common point and not $u$.

**Proof:** Suppose that $\gamma_{clt}(G) = 2$. Let $D = \{u, v\}$ be a $\gamma_{clt}(G)$ set of $G$. Then $D$ intersects every clique of $G$. Therefore, any clique of $G$ has either $u$ or $v$ as a vertex.

Conversely, suppose any two cliques of $G$ have either $u$ or $v$ as a common point satisfying the conditions in the hypothesis of the theorem. Then $\gamma_{clt}(G) \neq 1$. Therefore, $\gamma_{clt}(G) = 2$. But $\{u, v\}$ dominates every vertex of $V\setminus \{u, v\}$. Therefore, $\gamma_{clt}(G) = 2$.

**Corollary 2.3.** Let $T$ be a tree. Then $\gamma_{clt}(T) = 2$ if and only if $T$ is a double star.

**Proof:** From the above theorem 2.2, we get that $T$ has at least three edges and any two edges of $T$ have either $u$ as a common point or $v$ as a common point. Also there exists an edge of $T$ which is incident at $u$ only and not $v$ and an edge which is incident at $v$ only and not $u$. Therefore, $T$ is a double star. The converse is obvious.

**Remark 2.4.** The property of being a clt-set is super hereditary.

(i) If $D$ is a clt-set of $G$ and $u \in V(G) - D$, then $D \cup \{u\}$ is a clt-set of $G$.

(ii) Since clt-sets satisfy super hereditary property, a clt-set $D$ is minimal if and only if it is $1$-minimal.

**Theorem 2.5.** A clt-set $D$ is minimal if and only if, for every $u$ in $D$ one of the following holds:

(i) $u$ is an isolate of $D$.

(ii) There exists $v \in V-D$ such that $N(v) \cap D = \{u\}$.

(iii) There exists a clique $S$ of $G$ such that $S \cap D = \{u\}$.

**Proof:** Suppose $D$ is a minimal clt-set of $G$. Let $u \in D$. Then $D\setminus \{u\}$ is not a clt-set of $G$. Therefore, either $D\setminus \{u\}$ is not a dominating set of $G$ or $D\setminus \{u\}$ is not a clt-set of $G$. Hence, $u$ is an isolate of $D$ or there exists $v \in V-D$ such that $N(v) \cap D = \{u\}$ or there exists a clique $S$ of $G$ such that $(D\setminus \{u\}) \cap S = \phi$. That is, $S \cap D = \{u\}$. Conversely, suppose that $D$ is a clt-set of $G$ and for every $u$ in $D$ one of the above conditions holds. Then we have the following possibilities.

Suppose (i) holds, then $D\setminus \{u\}$ is not a dominating set of $G$.

Suppose (ii) holds, then also $D\setminus \{u\}$ is not a dominating set of $G$. 

Suppose (iii) holds, then \((D \setminus \{u\}) \cap S = \emptyset\). Therefore, \(D \setminus \{u\}\) is not a clt-set of \(G\). Therefore, \(D\) is a minimal clt-set of \(G\).

**Theorem 2.6.** \(\gamma_{clt}(K_{m,n}) = \min\{m, n\}\) for all \(m, n \geq 1\).

**Proof:** Let \(G = K_{m,n}\).

**Case (i):** \(m = 1, n = 1\).

Then obviously, \(\gamma_{clt}(K_{1,1}) = 1\).

**Case (ii):** \(m = 1, n \geq 2\).

In this case, \(G = K_{1,n}\) which is a star. Any edge of \(K_{1,n}\) is a clique. Therefore, \(\gamma_{clt}(K_{1,n}) = 1\).

**Case (iii):** \(m, n \geq 2\). Any clique of \(K_{m,n}\) is an edge. Therefore, \(\gamma_{clt}(K_{m,n}) = \min\{m, n\}\) for all \(m, n \geq 1\).

**Remark 2.7.** A maximal independent set of a graph \(G\) is a minimal dominating set of \(G\), but it need not be a clique transversal set of \(G\).

For example, let \(G = K_3^+\). Then the pendant vertices of \(G\) form a maximal independent set which is not a clique transversal.

**Figure 2:** The graph \(G = K_3^+\).

**Remark 2.8.** There is no relationship between \(\gamma_{clt}(G)\) and \(\beta_0(G)\). To illustrate this, consider the three graphs \(G_1\), \(G_2\) and \(G_3\) given in Figure 2.

\(\gamma_{clt}(G_1) = 1\) and \(\beta_0(G_1) = 2\). Therefore, \(\gamma_{clt}(G_1) < \beta_0(G_1)\).

\(\gamma_{clt}(G_2) = 1 = \beta_0(G_2)\). \(\gamma_{clt}(G_3) = 3\), \(\beta_0(G_3) = 2\) and \(\alpha_0(G_3) = 3\). \(\gamma_{clt}(G_3) > \beta_0(G_3)\).

**Figure 3:** Graphs \(G_1\), \(G_2\) and \(G_3\).
Theorem 2.9. Let $P$ be the Petersen graph. Then, $\gamma_{clt}(P) = 6$.

Proof: Let $P$ be the Petersen graph. $\gamma(P) = 3$, $\beta_0(P) = 4$ and $\omega_0(P) = 6$. Since any edge is a clique of $G$, $\gamma_{ch}(P) \geq \omega_0(P) = 6$. Let $S = \{v_2, v_4, v_5, v_6, v_7, v_8\}$. $S$ is a clique transversal dominating set of $P$. Therefore, $\gamma_{clt}(P) \leq |S| = 6$. Hence, $\gamma_{clt}(P) = 6$.  

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{petersen_graph}
\caption{Petersen graph.}
\end{figure}

Observation 2.10. $\gamma_{clt}(G_1 \cup G_2) = \gamma_{clt}(G_1) + \gamma_{clt}(G_2)$.

Observation 2.11. $\gamma_{clt}(G_1 + G_2) = \min\{\gamma_{clt}(G_1), \gamma_{clt}(G_2)\}$.

Proof: Assume that, $\gamma_{clt}(G_1) \leq \gamma_{clt}(G_2)$. Let $S$ be a $\gamma_{clt}$-set of $G_1$. Let $C$ be a clique of $G_1 + G_2$. Then $<V(C) \cap V(G_1)>$ is a clique of $G_1$ and $<V(C) \cap V(G_2)>$ is a clique of $G_2$. Since $S$ is a $\gamma_{clt}$-set of $G_1$, we have $S \cap (V(C) \cap V(G_2)) \neq \emptyset$. Therefore, $S \cap V(C) \neq \emptyset$. Hence, $S$ is a clique transversal set of $G_1 + G_2$. $S$ is a dominating set of $G_1 + G_2$ (since $S$ dominates $G_1$ and every vertex of $S$ is adjacent with every vertex of $G_2$). Therefore, $\gamma_{clt}(G_1 + G_2) \leq |S| = \min\{\gamma_{clt}(G_1), \gamma_{clt}(G_2)\}$.

References

