On felicitous labelings of $P_{r,2m+1}$, $P_{r}^{2m+1}$ and $C_n \times P_m$

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Abstract
A simple graph $G$ is called felicitous if there exists a one-to-one function $f : V(G) \rightarrow \{0,1,2, \ldots, q\}$ such that the set of induced edge labels $f^*(uv) = (f(u) + f(v)) \pmod{q}$ are all distinct. In this paper we show that $P_{r,2m+1}$, $P_{r}^{2m+1}$ and $C_n \times P_m$ are felicitous graphs.

Keywords: Labeling, felicitous labeling.

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1 Introduction
In this paper we consider only simple graphs. For notation and terminology, we refer to [2]. Lee, Schmeichel and Shee [7] introduced the concept of a felicitous graph as a generalization of a harmonious graph. A graph $G$ with $q$ edges is called harmonious if there is an injection $f : V(G) \rightarrow \mathbb{Z}_q$, the additive group of integers modulo $q$ such that when each edge $xy$ of $G$ is assigned the label $(f(x) + f(y)) \pmod{q}$, the resulting edge labels are all distinct. A felicitous labeling of a graph $G$, with $q$ edges is an injection $f : V(G) \rightarrow \{0,1,2, \ldots, q\}$ so that the induced edge labels $f^*(xy) = (f(x) + f(y)) \pmod{q}$ are distinct. Clearly, a harmonious graph is felicitous. An example of a felicitous graph which is not harmonious is the graph $K_{m,n}$, where $m, n > 1$.

Throughout this paper, $f$ denotes a 1-1 function from $V(G)$ to a subset of the set of non-negative integers and for any edge $e = xy \in E(G)$, $f^*(e) = f(x) + f(y)$.

In [4, 5, 6], Kathiresan introduced new classes of graphs denoted by $P_{a,b}$ and $P_{a}^b$ and discussed the magic labeling of $P_{a,b}$ [5]. In [8], the gracefulness of $P_{a,b}$ was discussed. It motivates us to discuss the felicitousness of the graphs $P_{a,b}$ and $P_{a}^b$.

2 Definitions and basic results

Definition 2.1. Let $u$ and $v$ be two fixed vertices. We connect $u$ and $v$ by means of $b \geq 2$ internally disjoint paths of length $a \geq 2$ each. The resulting graph embedded in a plane is denoted by $P_{a,b}$. Let $v_i^0, v_i^1, v_i^2, \ldots, v_i^a$ be the vertices of the $i^{th}$ copy of the path of length $a$, where $i = 1, 2, 3, \ldots, b$. $v_i^0 = u$ and $v_i^a = v$ for all $i$.

We observe that the graph $P_{a,b}$ has $(a - 1)b + 2$ vertices and $ab$ edges.

Definition 2.2. Let $a$ and $b$ be integers such that $a \geq 2$ and $b \geq 2$. Let $y_1, y_2, \ldots, y_a$ be the fixed vertices. We connect the vertices $y_i$ and $y_{i+1}$ by means of $b$ internally disjoint paths $P_i$ of length $i+1$
each, \(1 \leq i \leq a-1\) and \(1 \leq j \leq b.\) Let \(y_i, x_{i,j,1}, x_{i,j,2}, \ldots, x_{i,j,n} y_{i+1}\) be the vertices of the path \(P_i^j,\) \(1 \leq i \leq a-1\) and \(1 \leq j \leq b.\) The resulting graph embedded in a plane is denoted by \(P_a^b,\) where \(V(P_a^b) = \{y_i : 1 \leq i \leq a-j\} \cup \bigcup_{i=1}^{a-1} \bigcup_{j=1}^{b} \{x_{i,j,k} : 1 \leq k \leq i\}\) and \(E(P_a^b) = \bigcup_{i=1}^{a-1} \bigcup_{j=1}^{b} \{y_i x_{i,j,1} : 1 \leq j \leq b\} \cup \bigcup_{i=1}^{a-1} \bigcup_{j=1}^{b} \{x_{i,j,k} x_{i,j,k+1} : 1 \leq k \leq i-1\} \cup \bigcup_{i=1}^{a-1} \{x_{i,j,1} y_{i+1} : 1 \leq j \leq b\}.\)

We observe that the number of vertices of the graph \(P_a^b\) is \(\frac{ba(a-1)}{2} + a\) and the number of edges is \(\frac{b(a-1)(a+2)}{2}.\)

**Definition 2.3.** A subgraph \(H\) of a graph \(G\) is said to be an even subgraph of \(G,\) if the degree of every vertex of \(H\) is even in \(H.\)

**Result 2.4.** [1] An even subgraph of a felicitous graph with an even number of edges contains an even number of odd labelled edges.

**Result 2.5.** [1] No even graph with \(4n + 2\) edges is felicitous.

**Lemma 2.6.** [1] Let \(G\) be a graph with an odd number of edges and let \(f: V(G) \rightarrow \{0, 1, 2, \ldots, q\}\) be an odd edge labeling of \(G.\) Then, \(f\) is a felicitous labeling for \(G.\)

**Proof.** As \(f\) is an odd edge labeling of a graph \(G\) with odd number of edges, \(f(E(G)) = \{1, 3, 5, \ldots, 2q-1\}.\) After taking mod \(q, f(E(G)) = \{1, 2, 3, \ldots, q\}.\) So, \(f\) becomes felicitous labeling of \(G.\)

**Remark 2.7.** It is observed that as in Result 2.5, most of the even graphs are not felicitous. So, finding felicitous graphs with even number of edges is very difficult.

### 3 Main Results

**Theorem 3.1.** \(P_{r, 2m+1}\) is a felicitous graph for all values of \(m\) and for odd values of \(r.\)

**Proof:** Let \(u\) and \(v\) be the origin and the terminal vertices of the \((2m + 1)\) internally disjoint paths of length \(r\) in \(P_{r, 2m+1}.\) Let \(v_0^i, v_1^i, v_2^i, \ldots, v_m^i\) be the vertices of the \(i^{th}\) copy of the path, where \(i = 1, 2, 3, \ldots, 2m + 1,\) \(v_0^i = u\) and \(v_m^i = v\) for all \(i.\) The number of vertices of the graph \(P_{r, 2m+1}\) is \((r - 1)(2m + 1)+2\) and the number of edges is \((2m + 1)r.\)

It is enough to show that \(P_{r, 2m+1}\) admits odd edge labeling.

Define \(f\) on the vertex set of \(P_{r, 2m+1}\) as follows:

- \(f(u) = 0\)
- \(f(v) = (2m + 1)r\)

For \(1 \leq j \leq \frac{r-1}{2},\)

\[
f(v_{2j-1}^i) = (4m + 2)(j - 1) + (2i - 1), \quad 1 \leq i \leq 2m + 1,
\]

\[
f(v_{2j}^i) = \begin{cases} (6m + 2) + (4m + 2)(j - 1) - 4(i - 1), & 1 \leq i \leq m + 1 \\ 6m + (4m + 2)(j - 1) - 4(i - (m + 2)), & m + 2 \leq i \leq 2m + 1 \end{cases}
\]
Let $E_1 = \{ v^i_1 v^j_1 : 1 \leq i \leq 2m + 1 \}$.

$E_2 = \{ v^{m+1}_j v^{m+1}_{j+1}, v^m_j v^m_{j+1}, \ldots, v^1_j v^1_{j+1}, v^{2m+1}_j v^{2m+1}_{j+1}, v^2_m v^2_{j+1}, \ldots, v^{m+2}_j v^{m+2}_{j+1} : 1 \leq j \leq r - 2 \}$.

$E_3 = \{ v^{m+1}_{r-1} v^{m+1}_r, v^{2m+1}_{r-1} v^{2m+1}_r, v^m_{r-1} v^m_r, v^{2m}_{r-1} v^{2m}_r, \ldots, v^2_{r-1} v^2_r, v^{m+2}_r v^{m+2}_{r-1}, v^{1}_r v^{1}_{r-1} \}$.

The labels of the edges in $E_1$ are $2i - 1, 1 \leq i \leq 2m + 1$.

For $1 \leq j \leq r - 2$, the labels of the edges in $E_2$ are $2j (2m + 1) + 1, 2j (2m + 1) + 3, \ldots, 2j (2m + 1) - (2m - 1), 2j + 1(2m + 1) - (2m - 3), \ldots, 2j + 1(2m + 1) - 1$.

The labels of the edges in $E_3$ are $2(r - 1)(2m + 1) + 1, 2(r - 1)(2m + 1) + 3, \ldots, 2r(2m + 1) - 1$.

$f(E_1) = \{1, 3, 5, \ldots, 2(2m + 1) - 1\} = \{1, 3, 5, \ldots, 4m + 1\}$.

$f(E_2) = \{2(2m + 1) + 1, 2(2m + 1) + 3, \ldots, 4(2m + 1) - (2m - 1), 4(2m + 1) - (2m - 3), \ldots, 4(2m + 1) - 1, \ldots, 2(r - 2)(2m + 1) + 1, 2(r - 2)(2m + 1) + 3, \ldots, 2(r - 2)(2m + 1) - 1\} = \{4m + 3, 4m + 5, \ldots, 6m + 5, 6m + 7, \ldots, 8m + 3, \ldots, 2(r - 2)(2m + 1) + 1, 2(r - 2)(2m + 1) + 3, \ldots, 2(r - 1)(2m + 1) - 1\}$.

$f(E_3) = \{2(r - 1)(2m + 1) + 1, 2(r - 1)(2m + 1) + 3, \ldots, 2r(2m + 1) - 1\}$.

Now, $f(E(G)) = f(E_1) \cup f(E_2) \cup f(E_3)$.

$f(E(G)) = \{1, 3, 5, \ldots, 4m + 1, 4m + 3, 4m + 5, \ldots, 6m + 5, 6m + 7, \ldots, 8m + 3, \ldots, 2(r - 2)(2m + 1) + 1, 2(r - 2)(2m + 1) + 3, \ldots, 2(r - 1)(2m + 1) - 1, 2(r - 1)(2m + 1) + 1, 2(r - 1)(2m + 1) + 3, \ldots, 2r(2m + 1) - 1\} = \{1, 3, 5, \ldots, 2q - 1\}$.

Clearly, the above edge labelings are distinct and odd and hence $G$ admits odd edge labeling. Therefore, by Lemma 2.6, $P_r, 2m+1$ is a felicitous graph for all the values of $m$ and for odd values of $r$.

Example 3.2. A felicitous labeling of $P_{7,5}$ is shown in Figure 1.

![Figure 1: A felicitous labeling of $P_{7,5}$](image)

Corollary 3.3. $P_{a,b}$ is not a felicitous graph when $a \equiv 1(\text{mod}\ 2)$ and $b \equiv 2(\text{mod}\ 4)$.  


Proof. The number of edges of \( P_{a,b} = ab = (2k + 1)(4m + 2) = (8km + 4k + 4m + 2) = 4(2km + k + m) + 2 = 4l + 2 \) where \( l = 2km + k + m \) and \( l \in \mathbb{Z}^* \). Further, the graph \( P_{a,b} \) is even. Hence, \( P_{a,b} \) is not a felicitous graph when \( a \equiv 1(\text{mod } 4) \) and \( b \equiv 2(\text{mod } 4) \).

Theorem 3.4. \( P_{2m+1}^{2m+1} \) is a felicitous graph for all values of \( m \) and \( r \equiv 0, 3(\text{mod } 4) \).

Proof. Let \( y_1, y_2, \ldots, y_r \) be the fixed vertices. We connect the vertices \( y_i \) and \( y_{i+1} \) by means of \( 2m + 1 \) internally disjoint paths \( P_i^j \) of length \( i+1 \) each, \( 1 \leq i < r - 1 \) and \( 1 \leq j \leq 2m + 1 \). Let \( y_i, x_{i,j,1}, x_{i,j,2}, \ldots, x_{i,j,r}, y_{i+1} \) be the vertices of the path \( P_i^j \), \( 1 \leq i \leq r - 1 \) and \( 1 \leq j \leq 2m + 1 \). We observe that the number of vertices of the graph \( P_r^{2m+1} \) is \( \frac{(2m+1)(r(r-1))}{2} + r \) and the number of edges is \( \frac{(2m+1)(r(r-1))}{2} + r \).

It is enough to show that \( P_r^{2m+1} \) admits odd edge labeling. Define \( f \) on \( V(P_r^{2m+1}) \) as follows:

\[
f(y_i) = \left( \frac{i(i+1)}{2} - 1 \right)(2m+1), \quad 1 \leq i \leq r,
\]

\[
f(x_{i,j,1}) = 2j - 1, \quad 1 \leq j \leq 2m + 1,
\]

\[
f(x_{2,j,2}) = \begin{cases} 5(2m+1) - 1 - 4(j - 1), & 1 \leq j \leq m + 1 \\ 5(2m+1) - 3 - 4(j - (m + 2)), & m + 2 \leq j \leq 2m + 1 
\end{cases}
\]

For \( 1 \leq j \leq 2m + 1, \)

\[
f(x_{i,j,k}) = \begin{cases} f(x_{i-1,j,k}) + (2m+1)i & \text{if } k = 1,2 \text{ and } k + 1 \leq i \leq r - 1; \\ f(x_{i,j-1,k}) + (2m+1) & \text{if } k \text{ is odd, } 3 \leq k \leq r - 1 \text{ and } k \leq i \leq r - 1; \\ f(x_{i,j-2,k}) + (2m+1) & \text{if } k \text{ is even, } 4 \leq k \leq r - 1 \text{ and } k \leq i \leq r - 1.
\end{cases}
\]

Let \( E_1 = \{ y_1, x_{1,1}, x_{1,2}, x_{1,3}, \ldots, y_r, x_{r,2m+1} : 1 \leq i \leq r - 1 \} \), \( E_2 = \{ x_{i,m,k} x_{i,m+1,k+1}, x_{i,m,k} x_{i,m+1,k+1}, \ldots, x_{i,1,1} x_{i,1,k+1}, x_{i,2m-1,1} x_{i,2m+1,k+1}, x_{i,2m,1} x_{i,2m+1,k+1}, \ldots, x_{i,m+2,1} x_{i,m+2,k+1} : 2 \leq i \leq r - 1 \text{ and } 1 \leq k \leq i - 1 \} \),

\( E_3 = \{ y_1, x_{1,1}, y_{1,1}, x_{1,2}, y_{1,2}, x_{1,3}, y_{1,3}, \ldots, x_{r,2m+1}, y_{r,1} : 1 \leq i \leq r - 1 \} \) and \( E_4 = \{ x_{i,m+1,1} y_{i+1}, x_{i,2m+1,1} y_{i+1}, x_{i,m+1,1} y_{i+1}, x_{i,2m+1,1} y_{i+1}, x_{i,1,1} y_{i+1}, x_{i,2m,1} y_{i+1}, \ldots, x_{i,1,1} y_{i+1}, x_{i,2m,1} y_{i+1} : 1 \leq i \leq r - 1 \} \).

The edge labels of \( P^{2m+1}_r \) are as follows:

For \( 1 \leq i \leq r - 1 \), the labels of the edges in \( E_1 \) are \( (i(i+1) - 2)(2m+1) + 1, (i(i+1) - 2)(2m+1) + 3, \ldots, (i(i+1) - 2)(2m+1) + 2(2m+1) - 1 \).

For \( 2 \leq i \leq r - 1 \) and \( 1 \leq k \leq i - 1 \), the labels of the edges in \( E_2 \) are \( (i(i+1) - 2 + 2k)(2m+1) + 1, (i(i+1) - 2 + 2k)(2m+1) + 3, \ldots, (i(i+1) - 2 + 2k)(2m+1) + 2(2m+1) + 2, \ldots, (i(i+1) + 2k)(2m+1) - 1 \).

For \( 1 \leq i \leq r - 1 \) and \( i \equiv 1(\text{mod } 2) \), the labels of the edges in \( E_3 \) are \( (i(i+3) - 2)(2m+1) + 1, (i(i+3) - 2)(2m+1) + 3, \ldots, (i(i+3) - 2)(2m+1) + 2(2m+1) - 1 \).
For \( 1 \leq i \leq r - 1 \) and \( i \equiv 0 \pmod{2} \), the labels of the edges in \( E_4 \) are \((i(i + 3) - 2)(2m + 1) + 1, (i(i + 3) - 2)(2m + 1) + 3, \ldots , (i(i + 3) - 2)(2m + 1) + 2(2m + 1) - 1\) respectively.

Now, \( f(E(G)) = f(E_1) \cup f(E_2) \cup f(E_3) \cup f(E_4) = \{1, 3, 5, \ldots , 2(2m + 1) - 1, \ 4(2m + 1) + 1, \ 4(2m + 1) + 3, \ldots , 4(2m + 1) + 2(2m + 1) - 1, \ldots , (r(r - 1) - 2)(2m + 1) + 1, (r(r - 1) - 2)(2m + 1) + 3, \ldots \}
\( \cup \{6(2m + 1) + 1, 6(2m + 1) + 3, \ldots , 7(2m + 1), 7(2m + 1) + 2, \ldots , (r(r - 1) - 2) + 2(r - 2)(2m + 1) + 1, (r(r - 1) - 2) + 2(r - 2)(2m + 1) + 3, \ldots , (r(r - 1) + 2(r - 2))(2m + 1) - 1\} \cup \{2(2m + 1) + 1, 2(2m + 1) + 3, \ldots , 2(2m + 1) + 2(2m + 1) - 1, \ldots , (r - 1)(r + 2) - 2)(2m + 1) + 1, (r - 1)(r + 2) - 2)(2m + 1) + 3, \ldots , (r - 1)(r + 2) - 2)(2m + 1) + 1, (r - 1)(r + 2) - 2)(2m + 1) + 1\}
\( = \{1, 3, 5, \ldots , 2q - 1\} \).

Clearly, the above edge labelings are distinct and odd and hence \( G \) admits odd edge labeling. Therefore, \( P_{2m+1} \) is a felicitous graph for all values of \( m \) and \( r \equiv 0, 3 \pmod{4} \). \( \blacksquare \)

**Example 3.5.** A felicitous labeling of \( P_5 \) is shown in Figure 2.

![Figure 2: A felicitous labeling of \( P_5 \)](image)

**Corollary 3.6.** \( P_{a} \) is not felicitous when \( b \equiv 2 \pmod{4} \) and (i) \( a \equiv 0 \pmod{4} \) or (ii) \( a \equiv 3 \pmod{4} \).

**Proof.** (i) Let \( a \equiv 0 \pmod{4} \) and \( b \equiv 2 \pmod{4} \).

The number of edges of \( P_{a} \) is

\[
\begin{align*}
\text{The number of edges of } P_{a} &= \frac{b(a - 1)(a + 2)}{2} = \frac{(4k + 2)(4m - 1)(4m + 2)}{2} \\
&= \frac{2(2k + 1)(4m - 1)(4m + 2)}{2} \\
&= 2(8m^2 + 2m - 1)(2k + 1) \\
&= 2(16m^2k + 8m^2 + 4mk + 2m - 2k - 1)
\end{align*}
\]
\[ (1, 3, 2) \]

(ii) Let \( a \equiv 3 \pmod{4} \) and \( b \equiv 2 \pmod{4} \).

The number of edges of \( P_a^b \) is
\[
\frac{(4k + 2)(4m + 3 - 1)(4m + 3 + 2)}{2} = \frac{2(2k + 1)(4m + 2)(4m + 5)}{2} = 2(2k + 1)(2m + 1)(4m + 5)
\]
\[ = 2(16m^2k + 28km + 10k + 8m^2 + 14m + 5) \]
\[ = 4(8m^2k + 4m^2 + 14mk + 7m + 5k + 2) + 2 = 4l + 2 \quad \text{where } l = 8m^2k + 4m^2 + 14mk + 7m + 5k + 2 \text{ and } l \in \mathbb{Z}^+. \]

**Remark 3.7.** Let \( G \) be a \((p, q)\) graph. Let \( f \) be a felicitous labeling. Define \( f_1(uv) = f(u) + f(v) \) for every \( uv \in E(G) \). Then \( f^q(uv) = f_i(uv)(mod \ q) \).

**Theorem 3.8.** \( C_u \times P_m \) is felicitous for \( m \geq 4 \) and \( n \equiv 1 \pmod{2} \).

**Proof.** Case (i): when \( n = 3 \).

Let \( V(C_u \times P_m) = \{u_{ij} : 1 \leq i \leq 3 \text{ and } 1 \leq j \leq m\} \).

Define \( f : V(C_u \times P_m) \rightarrow \{0, 1, 2, \ldots, q = 6m - 3\} \) by
\[
f(u_{ij}) = \begin{cases} i - 1, & 1 \leq i \leq 3; \\ 3 + i, & 1 \leq i \leq 2 \text{ and } f(u_{23}) = 3; \\ 5 + i, & 1 \leq i \leq 3. 
\end{cases}
\]
\[
f(u_{ij}) = \begin{cases} f(u_{i(j-1)}), & 5 \leq j \leq m \text{ and } j \equiv 1 \pmod{2}; \\ f(u_{3(j-1)}) + \sigma_1(i), & 4 \leq j \leq m \text{ and } j \equiv 0 \pmod{2}; \end{cases}
\]
where \( \sigma_1(1) = 3 \).

Let \( E_1 = \{(u_{2j}u_{i1}), (u_{ij}u_{3j}), (u_{ij}u_{2j}) : 1 \leq j \leq m \text{ and } j \equiv 1 \pmod{2}\} \),
\( E_2 = \{(u_{12}u_{23}), (u_{23}u_{12}), (u_{12}u_{23}), (u_{23}u_{12}), (u_{ij}u_{3j}) : 4 \leq j \leq m \text{ and } j \equiv 0 \pmod{2}\} \) and
\( E_3 = \{(u_{11}u_{12}), (u_{12}u_{23}), (u_{23}u_{12}), (u_{12}u_{23}), (u_{i(j+1)}, (u_{1j}u_{3(j+1)}), (u_{3j}u_{3(j+1)}) : 3 \leq j \leq m - 1\}. \)

Now, \( E = E_1 \cup E_2 \cup E_3 \).

The labels of the edges in \( E_1 \) and \( E_2 \) are \( f_i(E_1) \cup f_i(E_2) = \{6j - 5, 6j - 4, 6j - 3 \mid 1 \leq j \leq m\} \).

The labels of the edges in \( E_3 \) are \( f_i(E_3) = \{6j - 2, 6j - 1, 6j \mid 1 \leq j \leq m - 1\} \).

Clearly, \( f(E(G)) = f_i(E_1) \cup f_i(E_2) \cup f_i(E_3) = \{1, 2, 3, \ldots, 6m - 6, 6m - 5, 6m - 4, 6m - 3\}. \)

After taking \( (mod \ q) \), \( f^q(E(G)) = f_i(E(G)) (mod \ q) = \{0, 1, 2, 3, \ldots, 6m - 5, 6m - 4\}. \)

**Case (ii):** when \( n \geq 5 \).
Let \( V(C_n \times P_m) = \{ u_{ij} : 1 \leq i \leq n \text{ and } 1 \leq j \leq m \} \).

Define \( f : V(C_n \times P_m) \to \{0, 1, 2, \ldots, q = (2m-1)n\} \) by
\[
f(u_{ij}) = i - 1, \quad 1 \leq i \leq n.
\]
Throughout this proof, addition being taken modulo \( n \) with residues 1, 2, 3, \ldots, \( n \).

Let \( \sigma_j = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ n-j+2 & n-j+3 & n-j+4 & \ldots & n-j+1 \end{pmatrix} \)
\[
f(u_{\sigma_j(i), j}) = n(j-2) + (n-1) + i, \quad 1 \leq i \leq n \quad \text{and} \quad 2 \leq j \leq m.
\]

The labels of the edges are,

For \( 1 \leq j \leq m \),
\[
f_1(u_{\sigma_j(i), j}, u_{\sigma_j(i+1), j}) = \begin{cases} n(2j-2) + (2i-1), & 1 \leq i \leq n-1 \\ n(2j-2) + (n-1), & i = n \end{cases}
\]
For \( 2 \leq j \leq m-1 \),
\[
f_1(u_{\sigma_j(i), j}, u_{\sigma_j(i), j+1}) = f(u_{\sigma_j(i), j}) = f(u_{\sigma_j(i), j}) = f(u_{\sigma_j(i), j}) = f(u_{\sigma_j(i+1), j+1}) \quad \text{by the definition of} \ \sigma_j.
\]

Therefore,
\[
f_1(u_{\sigma_j(i), j}, u_{\sigma_j(i+1), j}) = \begin{cases} n(2j-1) + (2i-1), & 1 \leq i \leq n-1 \\ n(2j-1) + (n-1), & i = n \end{cases}
\]

Let \( E_1 = \{ f_1(u_{\sigma_j(i), j}, u_{\sigma_j(i), j+1}) : 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq m \} \) and
\[
E_2 = \{ f_1(u_{\sigma_j(i), j}, u_{\sigma_j(i), j+1}) : 1 \leq i \leq n-1 \text{ and } 2 \leq j \leq m-1 \}.
\]

The labels of the edges in \( E_1 \) are,
\[
f_1(E_1) = \{ 1, 3, 5, 7, 9, \ldots, 2(n-1) - 1, n-1, 2n+1, 2n+3, \ldots, 2n + 2(n-1) - 1, 2n + n-1, \ldots, 2n(m-1) + 1, 2n(m-1) + 3, \ldots, 2n(m-1) + 2(n-2) - 1, 2n(m-1) + 2(n-1) - 1, 2n(m-1) + (n-1) \}.
\]
That is, \( f_1(E_1) = \{ 1, 3, 5, 7, 9, \ldots, 2n-3, n-1, 2n+1, 2n+3, \ldots, 3n-1, \ldots, 4n-3, \ldots, 2mn-2n+1, 2mn-2n+3, \ldots, 2mn-5, 2mn-3, 2mn-n-1 \}.
\]

The labels of the edges in \( E_2 \) are,
\[
f_1(E_2) = \{ 8, 10, 12, 14, 16, \ldots, 2(n-1) + 6, 2n-1, 3n+1, 3n+3, \ldots, 3n + 2(n-1) - 1, 3n + n - 1, \ldots, n(2(m-1) - 1) + 1, n(2(m-1) - 1) + 2(n-2) - 1, n(2(m-1) - 1) + 2(n-1) - 1, n(2(m-1) - 1) + (n-1) \} = \{ 8, 10, 12, 14, 16, \ldots, 2n-1, 2n+4, 3n+1, 3n+3, \ldots, 4n-1, \ldots, 5n-3, 2mn-3n+1, 2mn-3n+3, \ldots, 2mn-n-5, 2mn-n-3, 2mn-n-1 \}.
\]
\[
f_1(E_1) \cup f_1(E_2) = \{ 1, 3, 7, 8, 9, 10, \ldots, 2n-3, 2n-2, 2n-1, 2n, 2n+1, 2n+2, 2n+3, 2n+4, \ldots, 3n-1, 3n, 3n+1, 3n+2, \ldots, 4n-3, 4n-2, 4n-1, \ldots, 2mn-3n+1, 2mn-3n+2, 2mn-3n+3, \ldots, 2mn-n-3, 2mn-n-1 \}.
\]
\[ \ldots, 2mn - 2n - 1, 2mn - 2n, 2mn - 2n + 1, \ldots, 2mn - n - 3, 2mn - n - 2, 2mn - n - 1, 2mn - n, 2mn - 5, 2mn - 3 \].

After taking \((\text{mod } q)\), \(f^i(G(E(G))) = f_i(G(E(G))) \pmod{q} = \{1, 2, 3, \ldots, n - 3, n - 2, n - 1, n, n + 1, n + 2, \ldots, 2n - 1, 2n, 2n + 1, 2n + 3, \ldots, 3n - 1, 3n, 3n + 1, \ldots, 4n - 2, 4n - 1, n(2m - 3) + 1, n(2m - 3) + 2, \ldots, 2n(m - 1), 2n(m - 1) + 1, \ldots, n(2m - 1) - 1, n(2m - 1) \}.

Hence, \(C_n \times P_m\) is a felicitous graph for \(m \geq 1, n \geq 5\) and \(n \equiv 1 \pmod{2}\).

**Example 3.9.** A felicitous labeling of \(C_3 \times P_4\) is shown in Figure 3.

![Figure 3](image1.png)

**Figure 3:** A felicitous labeling of \(C_3 \times P_4\).

**Example 3.10.** A felicitous labeling of \(C_7 \times P_4\) is shown in Figure 4.

![Figure 4](image2.png)

**Figure 4:** A felicitous labeling of \(C_7 \times P_4\).

**References**

Sundaranar University, Tirunelveli, (1996), 47 - 61.


